D-collapsibility and its applications

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Abstract

A classical result of Helly [Hel23] asserts that if $\mathcal{K} = \{K_1, \dots K_n\}$ is a finite family of convex sets in $\mathbb{R}^d$ such that $\cap_{i \in I} K_i \neq \emptyset$ for all $I \subset [n]$ of cardinality $|I| \leq d+1$, then $\cap_{i=1}^{n} K_i \neq \emptyset$. Helly's theorem and its many extensions and generalizations form a central area of study in discrete geometry and its applications. A key ingredient that plays an important role in a variety of Helly type theorems is the notion of $d-$collapsibility. Let X be a simplicial complex. Pick $\sigma \in X$, assume that $|\sigma| \leq d$ and that it is contained in a unique facet τ. An elementary $d-$collapse is the operation $X \xrightarrow{[\sigma,\tau]} X - [\sigma, \tau]$ where $[\sigma, \tau] = \{f \in X : \sigma \subset f \subset \tau\}$, see for example Figure 3.1. We call a complex X $d-$collapsible if there exists a series of $d-$elementary collapses from it to $\emptyset$. Denote the collapse by $X \overset{d}{\rightsquigarrow} \emptyset$, and by $\mathcal{C}(X) = \min_{d \in \mathbb{N}} \left\{ X \overset{d}{\rightsquigarrow} \emptyset \right\}$. Recall that given $\mathcal{K}$ a family of sets, the nerve of $\mathcal{K}$, denoted by $N(\mathcal{K})$, is the simplicial complex with the vertex set $\mathcal{K}$, whose faces are $f \subset \mathcal{K}$ such that $\bigcap_{F \in f} F \neq \emptyset$. The link between $d-$collapsibility and convexity is the following key result of Wegner [Weg75]: If $\mathcal{K} = \{K_1, \dots, K_n\}$ is a family of convex sets in $\mathbb{R}^d$, then $N(\mathcal{K})$ is $d-$collapsible. Another central notion which is discussed in [Weg75] is the Leray number. Let the Leray number $\mathcal{L}(X)$ of X be the minimal d such that $\tilde{H}_i(Y, \mathbb{R}) = 0$ for all induced subcomplexes $Y \subset X$ and $i \geq d$. In [Weg75] Wegner showes that $\mathcal{C}(X) \geq \mathcal{L}(X)$ for any complex X.

In this thesis we study various aspects of $d-$collapsibility and their applications. In [MK07] Kalai and Meshulam show that given any two simplicial complexes X, Y the following inequalities hold:

$$\mathcal{L}(X \cap Y) \leq \mathcal{L}(Y) + \mathcal{L}(X)$$
$$\mathcal{L}(X \cup Y) \leq \mathcal{L}(Y) + \mathcal{L}(X) + 1$$

We try to reproduce their results for $d-$collapsibility. For the intersection inequality we show that:

Theorem 4.2. *Let X, Y be simplicial complexes, then*

$$\mathcal{C}(X \cap Y) \leq \mathcal{C}(Y) + \mathcal{C}(X).$$

For the union inequality we get a more restricted result. Recall that the star of the face $f \in X$ is the subcomplex $St(f, X) = \{\sigma \in X : f \cup \sigma \in X\}$. A simplicial complexes

X with the vertices V is $d-$star-collapsible if there exists is an order with the vertices $V = \{v_i\}_{i=1}^n$ for which, denoting $X_i := X\left[\{v_j\}_{j=i}^n\right]$, there is a $d-$collapse

$$X_i \overset{d}{\rightsquigarrow} St\left(v_i, X_i\right) \overset{d}{\rightsquigarrow} \emptyset.$$

With this new definition we are able to show:
Theorem 4.3. *Let X, Y be simplicial complexes. If Y is $\mathcal{C}(Y)-$star-collapsible then*

$$\mathcal{C}(X \cup Y) \leq \mathcal{C}(Y) + \mathcal{C}(X) + 1.$$

Finally we turn our discussion to a question posed and solved by Vorob'ev in [Vor62]:

Let X be a simplicial complex with the vertex set V. For each $v \in V$ let $\left(S_v, 2^{S_v}\right)$ be a finite measurable space. Denote $S = \prod_{v \in V} S_v$. For any $\tau \subset V$ let $(S, \mathcal{B}_\tau)$ be a finite measurable space, where:

$$\mathcal{B}_\tau = \left\{ A \times \prod_{v \in V \setminus \tau} S_v \ : \ A \subset \prod_{v \in \tau} S_v \right\}.$$

Denote by μ_τ a probabilty measure on $(S, \mathcal{B}_\tau)$. The probability measures μ_{τ_1} and μ_{τ_2} are consistent if for any $A \in \mathcal{B}_{\tau_1} \cap \mathcal{B}_{\tau_2}$, $\mu_{\tau_1}(A) = \mu_{\tau_2}(A)$. We call a family of probability measures $\{\mu_\tau\}_{\tau \in X}$ a consistent family if each two are. We call the family $\{\mu_\tau\}_{\tau \in X}$ extendable if there exists measure μ on $(S, \mathcal{B}_V)$ which is consistence with any μ_τ for $\tau \in X$. The question Vorob'ev posed and solved in [Vor62] is: whether a consistent family of probabilty measures on a simplicial complex is necessarily extendable? His solution is:
Theorem 5.1. (Vorob'ev [Vor62]) *A simplicial complex X is $1-$collapsible if and only if any consistent family of measures $\{\mu_\tau\}_{\tau \in X}$ on X is extendable.*

Since the notion of $d-$collapsibility did not yet exists when Vorob'ev wrote his paper, his original work used other concepts. Our contribution is translating his work into $d-$collapsibility 'language' and giving a small extension to his work due to the translation.

Abbreviations and Notations

$\mathbb{R}$	:	the real numbers
$\mathbb{N}$	:	the natural numbers
$[n]$	:	the set $\{1, 2, \ldots, n\}$
$\binom{V}{k}$	:	the collection of all subsets of size k of the set V
2^V	:	the collection of all subsets of the set V
$A \backslash B$	:	the subtraction of set B from the set A
$conv(A)$	:	the convex hull of A
$X - Y$	:	the subtraction of the family of sets Y from the family of sets X
$N(v)$	:	the neighbors of the vertex v
$G[A]$	:	the subgraph of G induced by A
$G(\mathcal{F})$	:	the intersection graph of the family $\mathcal{F}$
$u \overset{\gamma}{\rightsquigarrow} v$	:	the is a path γ from vertex u to v
$X(k)$	:	the collection of all k-dimensional simplices of the complex X
$X^{(k)}$	:	the k-dimensional skeleton of the complex X
$\dim(\sigma)$	:	the dimension of the simplex σ
$\dim(X)$	:	the dimension of the complex X
$X[U]$	:	the subcomplex of X induced by U
$St(\sigma, X)$	:	the star of the simplex σ in the complex X
$Lk(\sigma, X)$	:	the link of the simplex σ in the complex X
$X * Y$	:	the join of the complexes X and Y
$X \cap Y$	:	the intersection of the complexes X and Y
$X \cup Y$	:	the union of the complexes X and Y
$X \cong Y$	:	X is isomorphic to Y
$H_k(X)$	:	the $k-$th homology of X
$N(\mathcal{F})$	:	the nerve complex of the family $\mathcal{F}$
Δ_n	:	the $n-$simplex on vertex set $[n+1]$
$\partial\Delta_n$	:	the boundary of the $n-$simplex
$X(G)$	:	the clique complex of the graph G
$[\sigma, \tau]$	:	the set interval, $\{A : \sigma \subseteq A \subseteq \tau\}$
$X \xrightarrow{[\sigma,\tau]} X_\sigma$	:	the elementary collapse of σ in τ, from X to X_σ
$X \overset{d}{\rightsquigarrow} Y$	:	A $d-$collapse from X to Y
$\mathcal{K}(X)$	:	the $d-$ representability of X
$\mathcal{C}(X)$	:	the $d-$ collapsibility of X
$\mathcal{L}(X)$	:	the $d-$ the Leray number of X
$\mathcal{K}^d$	:	the family of all $d-$representable complexes
$\mathcal{C}^d$	:	the family of all $d-$collapsible complexes
$\mathcal{L}^d$	:	the family of all $d-$Leray complexes

Chapter 1

Introduction

Let us start by describing three types of graphs. Let $\mathcal{F}$ be a non empty family of sets. The *intersection graph* of $\mathcal{F}$, which we will denote by $G(\mathcal{F})$, is the graph whose vertices are the sets in $\mathcal{F}$, and there is an edge between two sets if and only if they intersect, i.e

$$E = \{\{S_1, S_2\} : S_1 \cap S_2 \neq \emptyset\}.$$

Note that any graph is isomorphic to some intersection graph. The second type of graphs, called *interval graphs*, are intersection graphs for a family of intervals in $\mathbb{R}$. The third type of graph is a chordal graph. We call a graph chordal if every cycle of length greater than 3 possess a chord (an edge between two of its vertices which is not a part of the cycle). A nice result by Lekkerkerker & Boland [LB62] gives us:

Lemma 2.2.5. (Lekkerkerker & Boland [LB62]) *Interval graphs are chordal.*

On the other hand, not all chordal graphs are interval graphs, an example can be found in Figure 2.2.

Chordal graphs have many equivalent definitions, one of which will be important for our discussion. For this we first need a couple of definitions:

Let $G = (V, E)$ be a graph. For $A \subset V$, denote by $G[A]$ the subgraph of G induced on the vertices A. A vertex $v \in V$ is called *simplicial* if its *neighborhood* $N(v) := \{u \in V : (v, u) \in E\}$ is a clique. Let $V = \{v_i\}_{i=1}^{n}$ be an order on the vertices of G. We call this ordering a *perfect elimination order* if v_j, for any $j \in [n]$, is a simplicial vertex in the graph $G\left[\{v_i\}_{i=j}^{n}\right]$.

Proposition 2.2.4. (Fulkerson & Gross [FG65]) *G is chordal if and only if it possesses a perfect elimination order.*

The importance of this characterization of chordal graphs will become clear soon.

The question which we will discuss now is: Are there higher dimensional extensions for the three types of graphs described above? To answer this we first need to define the notion of higher dimensional extension to graphs.

An *abstract simplicial complex*, or simply a *simplicial complex*, on a finite vertex set V, is a family $X \subset 2^V$ such that if $\tau \in X$ and $\sigma \subset \tau$ then $\sigma \in X$. A set $\sigma \in X$ is called a face. Inclusionwise maximal faces are called *facets*. Note that a simplicial complex is determined by its facets. The *dimension* of $\sigma \in X$ is $\dim \sigma := |\sigma| - 1$ and $dimX := \max\{\dim \sigma : \sigma \in X\}$. Let $X(i) := \{\sigma \in X : \dim \sigma = i\}$ and let $X^{(i)} = \bigcup_{j \leq i} X(j)$ denote the $i-$*skeleton* of X.

Now we are ready to define the high dimensional extensions. We begin with the intersection graphs. Let $\mathcal{F}$ be a family of sets. The *nerve* of $\mathcal{F}$, denoted by $N(\mathcal{F})$, is the simplicial complex with the vertex set $\mathcal{F}$, whose faces are $f \subset \mathcal{F}$ such that $\bigcap_{F \in f} F \neq \emptyset$. Note that the underline graph $N(\mathcal{F})^{(1)}$ of the nerve is exactly the intersection graph of $\mathcal{F}$, i.e. $(N(\mathcal{F}))^{(1)} = G(\mathcal{F})$. Like in the case of intersection graphs, any simplicial complex can be realized as the nerve of some family.

Recall that we obtain an interval graph by taking an intersection graph of a family of intervals in $\mathbb{R}$. Note that an interval in $\mathbb{R}$ is a convex set in $\mathbb{R}$. So for the extension of interval graphs we will take the nerves of a family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$. Such simplicial complexes are called $d-$*representable*. Denote by $\mathcal{K}^d = \{X : X \text{ is d-representable}\}$. We get an important property of $d-$representable complexes from Helly's theorem [Hel23].

Theorem 3.2. (Helly) *Let $K_1, K_2, \ldots, K_m$ be convex sets in $\mathbb{R}^d$. If $\bigcap_{i \in I} K_i \neq \emptyset$ for any $I \subset [m]$ of size $|I| \leq d+1$, then $\bigcap_{i \in [m]} K_i \neq \emptyset$.*

We get that $d-$representable complexes are completely determined by their $d-$skeleton.

In order to extend the notion of chordal graphs we will want to find a family of simplicial complexes that extend one of its characterization, and contains the family of $d-$representable complexes. Lets start with the characterizations, the one we want to extend is the perfect elimination.

A face $\sigma \in X$ is called *free* if it is contained in a unique facet of X. An *elementary $d-$collapse* in X is the operation $X \to X - [\sigma, \tau]$ where $\sigma \in X$ is a free face contained in the unique facet $\tau \in X$, and $|\sigma| \leq d$, and where

$$[\sigma, \nu] = \{\nu \in X : \sigma \subset \nu \subset \tau\}.$$

For an example of elementary $d-$collapse see Figure 3.1. A $d-$*collapsing* of X to a sub-simplicial complex X', denoted by $X \overset{d}{\rightsquigarrow} X'$, is a sequence :

$$X = X_1 \to X_2 \to \cdots \to X_{t-1} \to X_t = X',$$

where $X_i \to X_{i+1} = X - [\sigma_i, \tau_i]$ is an elementary $d-$collapse. Let

$$\mathcal{C}(X) := \min\{d : X \text{ is } d\text{−collepsible to } \emptyset\},$$

and denote the family of all $d-$collapsible complexes by $\mathcal{C}^d$. Note that, for any complex X, $\mathcal{C}(X) \leq dim(X)+1$, hence if a complex is finite so is its collapsibility number $\mathcal{C}(X)$.

As for the second part of the extension of chordal graphs we have:

Theorem 3.6. (Wegner [Weg75]) *Let $\mathcal{K}$ be a finite family of convex sets in $\mathbb{R}^d$, then $N(\mathcal{K})$ is $d-$collapsible.*

The inclusion $\mathcal{K}^d \subsetneq \mathcal{C}^d$ is strict. Tancer and Matousek showed in [MT09] that for any $d \in \mathbb{N}$, there exists a simplicial complex X such that $X \in \mathcal{C}^d$ but $X \notin \mathcal{K}^{2d-1}$.

Another family of simplicial complexes that arises in [Weg75] is the $d-$Leray family. Denote by $\tilde{H}_k(X, \mathbb{R})$ the reduced $k-$th homology of X. A simplicial complex X is $d-$Leray, denoted by $X \in \mathcal{L}^d$, if for all induced subcomplex $Y \subset X$ we have

$$\tilde{H}_i(Y, \mathbb{R}) = 0$$

for any $i \geq d$. The Leray number of a given complex is

$$\mathcal{L}(X) = \min\{d : X \text{ is } d\text{−Leray}\}.$$

This family is of interest to us because:

Theorem 3.8. (Wegner [Weg75]) *Any simplicial complex which is $d-$collapsible is also $d-$Leray, i.e. $\mathcal{C}^d \subset \mathcal{L}^d$.*

For $d = 1$ we get that $\mathcal{C}^1 = \mathcal{L}^1$ but the inclusion is strict for $d \geq 2$. Moreover in [MT09] the authors construct a family of simplicial complexes for which $\mathcal{C}(X) = 3d$ but $\mathcal{L}(X) = 2d$.

We now move our discussion to properties of $d-$collapsible complexes. In [MK07] it is shown that:

Theorem 1.1 (Meshulam & Kalai [MK07]). *For any pair of simplicial complexes X, Y:*

- $\mathcal{L}(X \cap Y) \leq \mathcal{L}(X) + \mathcal{L}(Y)$
- $\mathcal{L}(X \cup Y) \leq \mathcal{L}(X) + \mathcal{L}(Y) + 1$

Since $\mathcal{C}^d \subset \mathcal{L}^d$ it is a natural question to ask whether this inequalities persist for $d-$collapsible complexes as well. And this is exactly where our contribution starts.

For the intersection part of Theorem 1.1, we got:

Theorem 4.2. *For X, Y simplicial complexes*

$$\mathcal{C}(X \cap Y) \leq \mathcal{C}(X) + \mathcal{C}(Y).$$

An interesting application of the previous proposition is: Recall that for a pair of simplicial complexes X, Y with the vertex sets V_X, V_Y respectively. The *join* of X and Y is the simplicial complex with the vertex set $V_X \uplus V_Y$,

$$X * Y := \left\{\sigma_X \cup \sigma_Y \in 2^{V_X \uplus V_Y} : \sigma_X \in X, \sigma_Y \in Y\right\}.$$

Proposition 4.2.3. *Given X and Y simplicial complexes, then*

$$\mathcal{C}(X * Y) \leq \mathcal{C}(X) + \mathcal{C}(Y). \tag{1.1}$$

Surprisingly, the other direction of (1.1) is not trivial, and unfortunately unknown to us.

For the union part of Theorem 1.1, we unfortunately were unable to show that it translates to $d-$collepsibilty. But we were able to show a weaker version of it.

Recall that a *star* of a face f in the simplicial complex X is the subcomplex $St(f, X) = \{\sigma \in X : f \cup \sigma \in X\}$. X is *$d-$star-collapsible* if it has an order on the vertices $V = \{v_i\}_{i=1}^n$ for which, denoting $X_i := X\left[\{v_j\}_{j=i}^n\right]$, there is a $d-$collapse

$$X_i \overset{d}{\rightsquigarrow} St(v_i, X_i) \overset{d}{\rightsquigarrow} \emptyset.$$

The relation between the family of $d-$star-collapsible simplicial complexes and $d-$collapsible complexes is not fully understood. What we do know is that for any d, a $d-$star-collapsible simplicial complexes is $d-$collapsible and for $d = 1$:

Lemma 4.5.2. *Let X be a 1-collapsible simplicial complex on V. Then X is $1-$star-collapsible.*

For $d \geq 2$ the we currently do not know but we strongly believes that:

Conjecture 1.0.1. A $d-$collapsible complex X is *$d-$star-collapsible*

Using $d-$star-collapsibility we were able to get:

Theorem 4.3. *Let X, Y be simplicial complexes. If Y is* $\mathcal{C}(Y)-$star-collapsible, then*:*

$$\mathcal{C}(X \cup Y) \leq \mathcal{C}(X) + \mathcal{C}(Y) + 1.$$

Denote by Δ_n the $(n-1)$ −simplex. An interesting property of $d-$star-collapsibility which is in the heart of the proof of Theorem 4.3 is:

Proposition 4.3.2. *Let X be a simplicial complex with vertices set V. If X is $d-$star-collapsible, than there is a $(d+1)$-collapse $\Delta_{|V|-1} \overset{d+1}{\rightsquigarrow} X$.*

For $d=1$ we got a bit more:

Proposition 4.5.4. *Let X be a simplicial complex with the vertices V. X is 1-collapsible if and only if there is a 2-collapse $\Delta_{|V|-1} \overset{2}{\rightsquigarrow} X$.*

The last thing we want to discuss is an application of $d-$collapsibility:

Let X be a simplicial complex with the vertex set V. For each $v \in V$ let $(S_v, \mathcal{B}_v)$ be a finite measurable space. Denote by $S := \prod_{v\in V} S_v$. For any $\tau \subset V$ let $(S, \mathcal{B}_\tau)$ be a finite measurable space, where:

$$\mathcal{B}_\tau = \left\{ A \times \prod_{v\in V\setminus\tau} S_v \ : \ A \in \sigma\left(\prod_{v\in\tau} \mathcal{B}_v\right)\right\}.$$

Denote by μ_τ a probability measure on $(S, \mathcal{B}_\tau)$. The probability measures μ_{τ_1} and μ_{τ_2} are called *consistent* if $\mu_{\tau_1}(A) = \mu_{\tau_2}(A)$ for any $A \in \mathcal{B}_{\tau_1} \cap \mathcal{B}_{\tau_2}$. We call a family of probability measures $\{\mu_\tau\}_{\tau\in X}$ a *consistent family* if its measures are pairwise consistent. We call the family $\{\mu_\tau\}_{\tau\in X}$ *extendable* if there exists measure μ on $(S, \mathcal{B}_V)$ which is consistence with any μ_τ for $\tau \in X$.

We are interested in giving restrictions on the simplicial complex X such that we would get that any consistent family $\{\mu_\tau\}_{\tau\in X}$ is extendable. Vorob'ev showed in [Vor62] that:

Theorem 5.1. (Vorob'ev [Vor62]) *A simplicial complex X is $1-$collapsible if and only if any consistent family of measures $\{\mu_\tau\}_{\tau\in X}$ on X is extendable.*

Actually when Vorob'ev wrote his paper the notion of $d-$collapsibility did not exist yet. So we 'translated' the paper to 'modern language', and got a small extension to his ideas. Let Y be a *subcomplex of* X with a consistent family of measures $\{\mu_\tau\}_{\tau\in Y}$. We say that the family $\{\mu_\tau\}_{\tau\in Y}$ is *extendable to* X if there exist a consistent family of measures $\{\nu_\tau\}_{\tau\in X}$, such that $\nu_\sigma = \mu_\sigma$ for any $\sigma \in Y$. We show that:

Proposition 5.2.3. *Let X, Y be simplicial complexes. If $X \overset{2}{\rightsquigarrow} Y$ then any consistent family of measures on Y is extendable to X.*

Combining Proposition 4.3.2 and the previous proposition we obtain one direction of Theorem 5.1.

The thesis is organized as follows:

Chapter 2 contains the discussion about intersection, interval, and chordal graphs. It includes a proof that interval graphs are chordal, and a theorem characterizing chordal graphs.

Chapter 3 contains high dimensional 'extensions' to the first chapter. We give an introduction to simplicial complexes, and define nerve complexes, $d-$representability, $d-$collapsibility and $d-$Leray. We show the inclusion between the last three classes and some of their properties.

Chapter 4 is concerned with trying to prove Theorem 1.1 for $d-$collapsible complexes. We start by showing some known properties of $d-$collapsible complexes. Then we turn to show the intersection part of the Theorem 1.1, with which we show Proposition 4.2.3. We define $d-$star-collapsibility, which we later use to prove a weaker version for the union part of Theorem 1.1. We show that there is reason to believe that $d-$collapsible complexes are $d-$star-collapsible, And finish with showing that $1-$collapsible complexes are $1-$star-collapsible

Chapter 5 is dedicated to representation of the Theorem 5.1 in 'modern language', and providing a small extension of it.

Chapter 2

Graphs

2.1 Interval Graphs

Let $\mathcal{F}$ be a non-empty family of sets. The *intersection graph* of $\mathcal{F}$, which we will denote by $G(\mathcal{F})$, is the graph whose vertices are the sets in $\mathcal{F}$, and there is an edge between two sets if and only if they intersect, i.e.

$$E = \left\{ \{S_1, S_2\} \in F^2 : S_1 \cap S_2 \neq \emptyset \right\}.$$

Any graph $G{=}(V, E)$, can be realized as an intersection graph of a family of sets. For example, for any $u \in V$ let S_u be the set of all edges $e \in E$ containing u. Then the map $v \mapsto S_v$ is an isomorphism of G and $G\left(\{S_u\}_{u \in V}\right)$. Another, more efficient (in the sense of smallest size of $|\cup_{v \in V} S_v|$), construction was given in [EGP66], where $|\cup_{S \in V} S| \leq \frac{|V|^2}{2}$. In view of the fact that any graph is an intersection graph we can try to look at specific types of families of sets to get more interesting results. For example, an *interval graph* is an intersection graph for a family of intervals in $\mathbb{R}$, e.g the graph in Figure (2.1).

This class of graphs, as well as its extensions, have been well studied, see e.g.

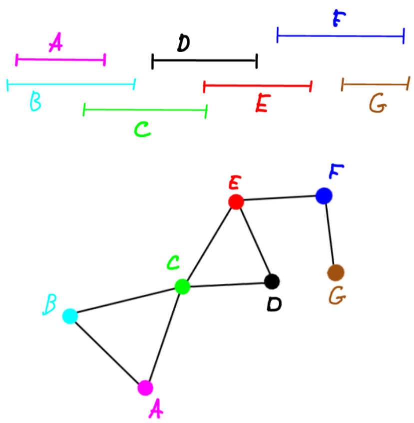


Figure 2.1: Interval graph example

[Eck93, FG65], and has multiple applications in biology [CS78, ZSF$^+$94], computer science [BNBYF$^+$01], etc. An alternative characterization for interval graphs is given by the following lemma:

Lemma 2.1.1. *G is an interval graph if and only if its maximal cliques can be ordered as $\{C_i\}_{i=1}^m$, such that $C_i \cap C_j \subset C_k$ for any $1 \leq i < k < j \leq m$*

Proof. First, assume that $G = (V, E)$ is an interval graph. Let $\mathcal{F} = \{I_v\}_{v \in V}$ be a family of intervals, such that $G(\mathcal{F}) = G$. Notice that if $C \neq C'$ are maximal cliques in G then:

$$\left(\bigcap_{v \in C} I_v\right) \cap \left(\bigcap_{v \in C'} I_v\right) = \emptyset.$$

Since otherwise either C or C' wouldn't have been maximal. Hence, we can order the maximal cliques, $C_1, \ldots, C_m$, according to $M(C_i) := \min\left(\bigcap_{v \in C_i} I_v\right)$, $i < j$ if and only if $M(C_i) < M(C_j)$. Let $1 \leq i < k < j \leq n$. If $v \in C_i \cap C_j$, then $[M(C_i), M(C_j)] \subset I_v$. Since $M(C_i) < M(C_k) < M(C_j)$ it follows that $M(C_k) \in I_v$, and the maximality of C_k implies that $v \in C_k$.

Conversely, assume that there is an ordering on the maximal cliques $\{C_i\}_{i=1}^m$ of G, such that $C_i \cap C_j \subset C_k$ for $1 \leq i < k < j \leq n$. For each $v \in V$ consider the interval

$$I_v = [\min\{i : v \in C_i\}, \max\{j : v \in C_j\}].$$

Denote $\hat{G} = G(\{I_v\}_{v \in V})$. We claim that $\hat{G} \cong G$. Let $f : V \to \{I_v\}_{v \in V}$ where $f(v) = I_v$, f is 1-1 and onto. We are left to show that any edge in G is mapped to an edge in $\hat{G}$. Now, on one hand for any $(u, v) \in E$, there exists a clique C_i which contains it. Thus $i \in I_v \cap I_u$ which gives us that $(f(u), f(v)) \in \hat{E}$. On the other hand, if $(u, v) \in \hat{E}$ then $I_u \cap I_v \neq \emptyset$. Since for every interval both of its boundaries are integers, any intersection is an interval with its boundary's being integers. Hence there exists an integer $i \in I_v \cap I_u$, which tells us that $u, v \in C_i$ and therefore $\left(f^{-1}(u), f^{-1}(v)\right) \in E$. Hence f is an isomorphism and $\hat{G} \cong G$. ■

2.2 Chordal Graphs

Another interesting class of graph, is the one of *chordal graphs.* A graph $G = (V, E)$ is called chordal if every cycle of length greater than 3 possess a chord, an edge between two of its vertices which is not part of the cycle, for example see Figure 2.1. The notion of chordality has several equivalent definitions. We aim to describe some of them in the following discussion. First, a small but useful observation that will help us later on. Given $G = (V, E)$ and $S \subset V$ the *induced subgraph on S is $G[S] = (S, E(S))$ where* $E(S) := \{e \in S \times S : e \in E\}$.

Claim 2.2.1. *If G is chordal then for any $S \subset V$ its induced sub graph $G[S]$ is also Chordal.*

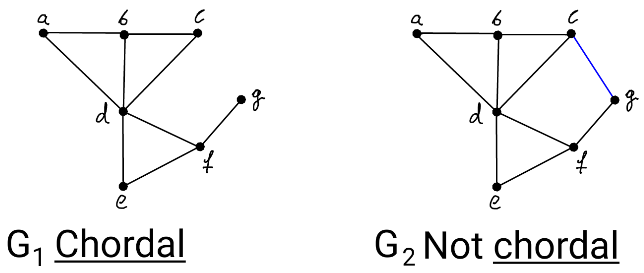


Figure 2.1: Chordal graph example

Proof. Any cycle $|C| > 3$ in G' also exists in G, and therefore has a chord e in G. As both ends of the chord e are in S, it follows that e is a chord for C in G'. ■

Let $G = (V.E)$ be a graph, and let $a, b \in V$. An $a - b$ *separator* is a subset $S \subset V$ such that $a, b \notin S$ and a and b belong to two different connected components of $G[V - S]$. An $a - b$ separator is minimal if there is no proper subset $S' \subsetneq S$ such that S' is a $a - b$ separator. For example, in Figure 2.1 the minimal $c - f$ separator in G_1 and G_2 are, respectively, $\{d\}$ and $\{d, g\}$. The following is a result of Dirac [Dir61], is a characterization of chordality:

Proposition 2.2.2 (Dirac [Dir61]). *G is a chordal graph if and only if every minimal $a - b$ separator induces a clique in G.*

Proof. Let $G = (V, E)$ be a chordal graph, $a, b \in E$ and $S \subset V$ a minimal $a - b$ separator. If $|S| < 2$, it is either an empty set or a single vertex and hence a clique or it dosent exist. Hence we assume that $|S| \geq 2$. Let A, B be the connected components of the vertices a, b respectively in $G\left[V \backslash S\right]$. By the minimality of S, for any $x, y \in S$ there is a minimal path $p_A = [x \to a_1 \to \cdots \to a_t \to y]$ where $a_i \in A$, since there is a path from both x and y to a. Similarly there is a minimal path $P = [y \to b_1 \to \cdots \to b_s \to x]$ where $b_i \in B$. Combine both paths to get a cycle $C = [x \to a_1 \to \cdots \to a_t \to y \to b_1 \to \cdots \to b_s \to x]$. Since G is chordal and the length of the cycle is grater than 4, there exists a chord in C. There can not be an edge between (a_i, b_j) since they are separated by S. By the minimality of P_A and P_B, the graph G contains no edges of the form:

$$\begin{aligned}
&\{x, a_i\}, i > 1\,; \{a_i, y\}, i < t\,; \\
&\{y, b_i\}, i > 1\,; \{b_i, x\}, i < s\,; \\
&\{x_i, x_j\}, \{y_i, y_j\}, |j - i| > 1
\end{aligned}$$

It follows that the chord is $\{x, y\}$. As this holds for any $x, y \in S$, it follows that S is a clique.

Now, let G be a graph where every minimal $a - b$ separator induces a clique. For a cycle $[v_0 \to a \to v_1 \to \cdots \to v_t \to b]$ with $t \geq 1$. Any $a - b$ separator must contain at least two vertices from this cycle for which their edge is not part of the cycle, but the would mean that there is a chord in the cycle. And hence G is chordal. ■

A vertex v of $G = (V, E)$ is called *simplicial* if its *neighborhood* $N(v) := \{u \in V : (v, u) \in E\}$ is a clique. Dirac showed in [Dir61] that a chordal graph always has a simplicial vertex:

Lemma 2.2.3 (Dirac [Dir61])**.** *Every chordal graph $G = (V, E)$ has a simplicial vertex. If the graph is not the complete graph, it has two nonadjacent simplicial vertexes.*

Proof. Let G be a chordal graph. We show this lemma by induction on $|V|$. For $|V| = 1$, G is just a vertex and we are done.

Let $|V| = n$ and assume we showed the lemma for any $|V| < n$. On one hand, if G is a complete graph then any vertex is simplicial. On the other hand, if G is not complete there are at least two nonadjacent vertices, denote them by $a, b \in V$. Let $S \subset V$ be a minimal $a - b$ separator. Denote by A, B the connected component of a, b in $G[V \backslash S]$ respectively. By Claim 2.2.1 the induced graph on $G[S \cup A]$ is chordal. By induction $G[S \cup A]$ either has two nonadjacent simplicial vertices which means that at least one of them is not in S. Else, G is a complete graph and then all of its vertices are simplicial and again and at least one of them is not in S. Denote the simplicial vertex which is not is S by v_A. Note that v_A is also a simplicial vertex in G, since its neighborhood is the same in G and in $G[S \cup A]$. The same procedure on $G[S \cup B]$ produces another simplicial vertex $v_B \in B/S$. Since both v_A and v_B are not in S it follows that they are not adjacent and therefore we have two nonadjacent simplicial vertexes. ■

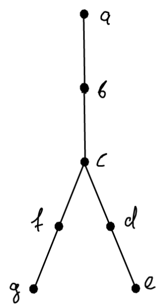
Given a graph $G = (V, E)$ and an order on the vertices $V = \{v_i\}_{i=1}^{n}$. We call this ordering a *perfect elimination order* if v_j, for any $j \in [n]$, is a simplicial vertex in the induced graph on $\{v_i\}_{i=j}^{n}$. For example, in Figure 2.1 G_1 has a perfect elimination order $[g, f, e, c, d, b, a]$. G_2 however does not have one. Accordingly to this definition we get another characterization of chordality by Fulkerson & Gross [FG65].

Proposition 2.2.4 (Fulkerson & Gross[FG65])**.** *G is chordal if and only if it possesses a perfect elimination order*

Proof. Assume that $G = (V, E)$ is chordal. We argue by induction on $|V|$ that G has a perfect elimination ordering. The case $|V| = 1$ is clear.

Now assume that $|V| = n > 1$. Since G is chordal by Lemma 2.2.3 we know that there is a simplicial vertex $v \in V$, which means that $N(v)$ is a clique. According to Claim 2.2.1, the graph $G[V - \{v\}]$ is a chordal graph, with $n - 1$ vertices, which by induction has a perfect elimination order $[u_1, u_2, \dots u_{n-1}]$. Thus $[v, u_1, u_2, \dots u_{n-1}]$ is an elimination order for G.

For the other direction, assume that $v_1, \dots, v_n$ is a perfect elimination order of G. We prove by induction on $|V|$ that G is chordal. The case $|V| = 1$ is clear.



[H]

Figure 2.2: Chordal graph but not Interval

Assume that $|V| = n > 1$. By induction we know that $G\left[\{v\}_{i=2}^{n}\right]$ is a chordal graph. It remains to show that any cycle C, with length greater than 3, has a chord. If $v_1 \notin C$ then it has the came chord as in $G\left[\{v\}_{i=2}^{n}\right]$. If $v_1 \in C$, let u, w be the neighbors of v_1 in C. As v_1 is a simplicial vertex it follows that $\{u, w\} \in E$, thus C has a chord. ■

We make a quick stop from the characterization of chordality. In order to talk about the connection of Interval graphs and chordality.

Lemma 2.2.5 (Lekkerkerker & Boland[LB62])**.** *Interval graphs are chordal*

Proof. We show the lemma by induction on $|V| = n$. For $n = 1$ it is clear.

Let $G = ([n], E)$ be the interval graph associated with the intervals

$$I_1 = [a_1, b_1], \ldots, I_n = [a_n, b_n].$$

Choose $1 \leq i \leq n$ such that $a_i = \max\{a_j : j \in [n]\}$. We claim that $v_1 := i$ is a simplicial vertex in G. Indeed, let $k, l \in N(i)$, then $I_k \cap I_i \neq \emptyset$ and $I_l \cap I_i \neq \emptyset$. By the choice of i, it follows that $a_i \in I_k \cap I_l$ and thus $\{k, l\} \in E$. Thus $N(i)$ is a clique. As $G\left[[n] \setminus \{i\}\right]$ is still an interval graph it follows by induction that it has an perfect elimination order $v_2, \ldots, v_n$. Therefore $v_1, \ldots, v_n$ is a perfect elimination order of G. ■

Note that, the inclusion in Lemma 2.2.5 is strict, i.e. there are chordal graphs which are not interval. For example, see the graph G in Figure 2.2. On one hand, there are no cycles so clearly G is chordal. On the other hand, If we look at the intervals $I_a \cup I_b, I_g \cup I_f$, and $I_e \cup I_d$ they can not intersect since there are no edges between these vertices. None of them is contained in I_c but all of them intersect with I_c. Hence each of them contains at least one of the boundary points of I_c. But since there are 2 boundary points in I_c there are at least two of them containing the same point. But that would mean that they would have an edge in the graph which they do not.

Coming back to characterization of chordal graphs. Since any graph is an intersection graph, one can ask whether there is a nice class of families for which their intersection graphs are exactly the chordal graphs.

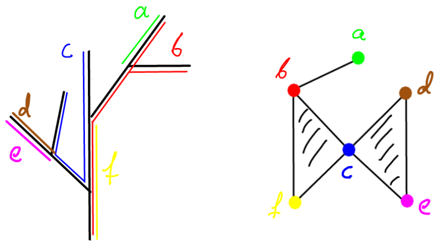
[b] 

Figure 2.3: Sub-graph tree

A *sub-tree graph* is the intersection graph of a family $\mathcal{F}$ of sub-trees of some tree T, i.e.:

$$\mathcal{F} \subset \left\{ \hat{T} \subset T : \hat{T} \text{ is a tree} \right\}.$$

For example see Figure 2.3. In [Gav74] Gavril showed that sub-tree graphs and chordal graph are one and the same. But first a small lemma that will help us later:

Lemma 2.2.6 (Gavril [Gav74])**.** *Let $\{C_i\}_{i=1}^n$ be the maximal cliques of the graph $G = (V, E)$. G is a sub-tree graph if and only if there exists a tree T with vertices $\{C_i\}_{i=1}^n$ such that for any $v \in V$ the induced subgraph $T[\mathcal{C}^v]$, where $\mathcal{C}^v := \{C_i : v \in C_i\}$, is connected.*

Proof. Assume that G is a sub-tree graph, hence there exist a family of sub-trees $\mathcal{F} = \{T_v\}_{v \in V}$ of a tree T such that $G(\mathcal{F}) \cong G$. Denote the maximal cliques if G by $\{C_i\}_{i=1}^n$. For any maximal clique C_i in G note that $S_i := \bigcap_{v \in C_i} T_v \neq \emptyset$ and that $S_i \cap S_j = \emptyset$ for any $j \neq i$. Pick a vertex $v_i \in S_i$ for any $i \in [n]$. Now let $T_C = (\{C_i\}_{i=1}^n, E_C)$ be a graph, where the edge $(C_i, C_j) \in E_C$, if and only if there exist simple path $v_i \overset{\gamma}{\rightsquigarrow} v_j$ in T such that $s_k \cap \gamma = \emptyset$, for any $k \neq i, j$. T_C is a tree since a cycle in T_C induces a cycles in T. For any $C_j, C_i \in \mathcal{C}^u$ and $u \in V$, there is a simple path γ between any v_i and v_j in T_u and hence there is a path between C_j and C_i in $T_C[\mathcal{C}^u]$.

Now, assume that there is a tree T_C on the vertices $\{C_i\}_{i=1}^n$, such that $T[\mathcal{C}^v]$ is connected for any $v \in V$. Pick the family $\mathcal{F} = \{\mathcal{C}^v : v \in V\}$ of sub trees in T, and let $G' := G(\mathcal{F})$ be the sub-tree graph. We now show that the map $v \mapsto \mathcal{C}^v$ is an isomorphism. Since it is 1-1 and onto all that is left to show is that any edge $\{v, u\}$ is in G if and only if the edge $\{\mathcal{C}^v, \mathcal{C}^u\}$ is in G'. On one hand, for any edge in G', $\{\mathcal{C}^v, \mathcal{C}^u\}$, we know that $\mathcal{C}^v \cap \mathcal{C}^u \neq \emptyset$. Hence for some $k \in [n]$, there exists a maximal clique C_k for which which $v, u \in C_k$, therefore $\{u, v\} \in E$. On the other hand if $(u, v) \in E$, then there is a maximal clique C that contains both of them, but that would mean that $\mathcal{C}^v \cap \mathcal{C}^u \neq 0$ and hence $\{\mathcal{C}_v, \mathcal{C}_u\}$ is an edge in G'. Therefore $G \cong G(\mathcal{F})$ and hence G is a sub-tree graph. ∎

Proposition 2.2.7 (Gavril [Gav74])**.** *G is a chordal Graph if and only if G is a sub-tree graph.*

Proof. Let $G = (V, E)$ be a sub-tree graph i.e. G is an intersection graph of a family of sub-trees $\mathcal{F} = \{T_v\}_{v \in V}$ of a tree T. Assume for contradiction that G is not chordal, hence we have a chordless cycle C with $|C| = n \geq 4$. Denote the cycle by:

$$C = v_1 \to v_2 \to \cdots \to v_{n-1} \to v_n.$$

Now pick $x_1 \in T_{v_1} \cap T_{v_n} \neq \emptyset$. Since $T_{v_1} \cap T_{v_2} \neq \emptyset$ there exists a simple paths which lies entirely in T_{v_1} from x_1 to any vertex in $T_{v_1} \cap T_{v_2}$. Pick $x_2 \in T_{v_1} \cap T_{v_2}$ with the shortest path $x_1 \overset{\gamma_1}{\rightsquigarrow} x_2$. Similarly we get a vertex $x_i \in T_{v_i}$ and $x_i \overset{\gamma_i}{\rightsquigarrow} x_{(i+1) \mod n}$ where the entire path lies entirely inside T_{v_i}. Combining all the paths we get, the path $\gamma \in T$:

$$\gamma := x_1 \overset{\gamma_1}{\rightsquigarrow} x_2 \overset{\gamma_2}{\rightsquigarrow} \dots x_{n-1} \overset{\gamma_{n-1}}{\rightsquigarrow} x_n \overset{\gamma_n}{\rightsquigarrow} x_1$$

For any $|i - j \pmod n)| \geq 2$ the paths $\gamma_i \cap \gamma_j = \emptyset$ since otherwise there would have been a chord in C. We get that γ is a cycle in T, which is a contradiction to T being a tree.

Now on the other hand assume that $G = (V, E)$ is chordal. We argue by induction on the size of $|V|$ that G is sub-tree graph. For $|V| = 1$, it is clear. Let $|V| = n$ and assume we already proved the claim for $|V| < n$. From Lemma 2.2.3 there is a simplicial vertex $v \in V$. From Lemma 2.2.1 the graph $G' := G[V \setminus \{v\}]$ is chordal, as a induced subgraph of a chordal graph G. Since $|V \setminus \{v\}| = n - 1$ by the induction hypothesis we know that G' is sub-tree graph. From Lemma 2.2.6 there is a tree $T = (\{C_i\}_{i=1}^m, E_T)$, where C_i are maximal cliques in G' and for any $\mathcal{C}^u := \{C_i : u \in C_i\}$ the induced tree $T[\mathcal{C}^u]$ is connected for any $u \in V / \{v\}$. Denote by $S = N_G(v) \cup \{v\}$. We now split the proof to 2 cases:

1. $N(v)$ is a maximal clique in G': Without loss of generality assume that $C_1 = N(v)$. Denote by $\hat{T} = \left(\hat{V}, \hat{E}_T\right)$ where the vertex set is $\hat{V} = S \cup \{C_i\}_{i=2}^m$ and $e \in \hat{E}_T$ if and only $e \in E_T$ or if $e = \{S, C_i\}$ and $\{C_1, C_i\} \in E_T$. We get a tree for which any $u \in V \setminus \{v\}$ $\hat{T}[\mathcal{C}^u]$ is connected, since T is. $\hat{T}[\mathcal{C}^v]$ is a graph on one vertex hence also connected. Therefore by Lemma 2.3 G is a sub-tree graph.

2. $N(v)$ is not a maximal clique in G': Since $N(v)$ is not a maximal clique in G' there is a maximal clique containing it without loss of generality assume it is C_1. Denote by $\hat{T} = \left(\hat{V}, \hat{E}_T\right)$ where the vertex set is $\hat{V} = S \cup \{C_i\}_{i=1}^m$ and $\hat{E} := E \cup \{\{S, C_1\}\}$. We get a tree that for any $u \in V \setminus N(v)$ $\hat{T}[\mathcal{C}^u]$ is connected since T is. For any $u \in N(v)$, $\hat{T}[\mathcal{C}^u]$ is connected since it is a sub tree of T with an extra edge $\{C_1, S\}$, and $\hat{T}[\mathcal{C}^v]$ is a graph on one vertex hence also connected. Therefore by Lemma 2.3 G is a sub-tree graph. ■

Summarizing the results of this section we state the following theorem:

Theorem 2.1. *Let $G = (V, E)$ be a graph, then the following are equivalent:*

1. *G is a chordal graph,*
2. *Every minimal $a - b$ separator in G induces a clique,*
3. *G has a perfect elimination order,*
4. *G is a sub-tree graph.*

Chapter 3

Higher dimensions

3.1 Simplicial Complex & Nerves

In this chapter we introduce high dimensional extensions of some of the graph theoretical concepts considered in chapter 2. We first define the high dimensional analogue of a graph. An *abstract simplicial complex*, or simply a *simplicial complex*, on a finite vertex set V, is a family $X \subset 2^V$ such that if $\tau \in X$ and $\sigma \subset \tau$ then $\sigma \in X$.

A set $\sigma \in X$ is called a face. Inclusionwise maximal faces are called *facets.* Note that a simplicial complex is determined by its facets. The *dimension* of $\sigma \in X$ is $\dim \sigma := |\sigma| - 1$ and $dimX := \max\{\dim \sigma : \sigma \in X\}$. Let $X(i) := \{\sigma \in X : \dim \sigma = i\}$ and let $X^{(i)} = \bigcup_{j \le i} X(j)$ denote the $i-$*skeleton* of X. The $(n-1)-$simplex with the vertex set $V = [n]$ is the simplicial complex $\Delta_{n-1} := 2^V$. Its boundary is $\partial\Delta_{n-1} = \{\sigma \subset [n] : \sigma \neq [n]\}$. The simplicial complex Y is a *subcomplex* of X if $Y \subset X$. For $A \subset V$, let $X[A] = \{\sigma \in X : \sigma \subset A\}$ be the *induced subcomplex* of X on the vertex set A.

The *star* of a face $f \in X$ is the subcomplex:

$$St(f, X) = \{\sigma \in X : f \cup \sigma \in X\},$$

and the *link* of a face $f \in X$ is the subcomplex:

$$Lk(f, X) = \{\sigma \in X : f \cup \sigma \in X, f \cap \sigma = \emptyset\}.$$

We next define the high dimensional extension of intersection graphs. Let $\mathcal{K}$ be a family of sets. The *nerve* of $\mathcal{K}$, denoted by $N(\mathcal{K})$, is the simplicial complex with the vertex set $\mathcal{K}$, whose faces are $f \subset \mathcal{K}$ such that $\bigcap_{F \in f} F \neq \emptyset$, see for example Figure 3.1 .

Note that the underline graph $N(\mathcal{K})^{(1)}$ of the nerve is exactly the intersection graph of $\mathcal{K}$, i.e. $(N(\mathcal{K}))^{(1)} = G(\mathcal{K})$.

Any simplicial complex $X \subset 2^V$ can be realized as the nerve of some family $\mathcal{K}$. For

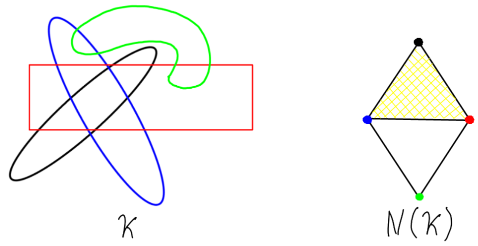


Figure 3.1: Nerve of the family $\mathcal{K}$

example, for any $v \in V$ let

$$K_v = \{f : v \in f;\ f \text{ is a facet of } X\}$$

and let $\mathcal{K} = \{\mathcal{K}_v\}_{v \in V}$. Then the map $v \mapsto K_v$ is an isomorphism of X and $N(\mathcal{K})$.

We briefly recall the definition of simplicial homology. For a detailed account, see e.g. Chapter 2 in [Hat01].

Let X be a simplicial complex and let $\mathbb{F}$ be a fixed field. The space of *$k-$chains* of X over $\mathbb{F}$, denoted by $C_k(X, \mathbb{F})$, is the vector space with generators $[v_0, .., v_k]$ where $\{v_0, .., v_k\} \in X(k)$, modulo the relations

$$\left(\left[v_{\pi(o)}, ..., v_{\pi(k)}\right] = sgn(\pi)\,[v_0, ..., v_k]\right),$$

where π is any permutation on V and $sgn(\pi)$ is it's parity. Let $\partial_k : C_k(X, \mathbb{F}) \to C_{k-1}(X, \mathbb{F})$ be the *boundary homomorphism,* which is defined on the generators of $C_k(X, \mathbb{F})$ as follows:

$$\partial_k([v_0, ..., v_k]) = \sum_{i=0}^{k} (-1)^i [v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k].$$

Let $Z_k := \ker \partial_k \subset C_k(X, \mathbb{F})$ be the space of $k-$cycles. Let $B_k = \partial_{k+1}(C_{k+1}(X, \mathbb{F}))$ be the space of $k-$boundaries in X. A basic computation shows that $\partial^2 = 0$, and hence $B_k \subset Z_k$. The $k-$th homology of X is:

$$H_k = {}^{Z_k}/_{B_k}$$

3.2 The $d-$Representable Property

Now since we have an analog for intersection graphs, we can define an analog for interval graphs. First, recall that the set $K \subset \mathbb{R}^d$ is a *convex set* if for any two points $x, y \in K$ and for any $t \in [0, 1]$

$$tx + (1 - t)\,y \in K.$$

A simplicial complex X is *$d-$representable* if there exists a family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$, such that the simplicial complex $N(\mathcal{F})$ is isomorphic to X. Denote by $\mathcal{K}^d$ the family of $d-$representable complexes. Observe that intervals are simply convex sets in $\mathbb{R}^1$, and hence for a given family $\mathcal{F}$ of intervals in $\mathbb{R}$, an interval graph is the underlying graph of $N(\mathcal{F})$. To explore the notion of $d-$representability of simplicial complexes further we will need a few results from discrete geometry.

Lemma 3.2.1. Let $A \subset \mathbb{R}^d$. *The following conditions are equivalent:*

1. *A is convex*

2. *For any $n \in \mathbb{N}$, if $a_i \in A$, $\lambda_i \geq 0$ for any $i \in [n]$, and $\sum_{i=1}^{n} \lambda_i = 1$ then $\sum_{i=1}^{n} \lambda_i a_i \in A$*

Proof. (2) $\implies$ (1) Since $\sum_{i=1}^{n} \lambda_i a_i \in A$ is true for $n = 2$, we get that A is convex since it is the definition of being convex.

(1) $\implies$ (2) We show this by induction on n. For $n = 2$ it is true since A is convex. Now take $\sum_{i=1}^{k} \lambda_i a_i$. Denote by $\lambda := \sum_{i=1}^{k-1} \lambda_i$ and by induction we get that

$$\frac{1}{\lambda} \sum_{i=1}^{k-1} \lambda_i a_i \in A.$$

Since A is convex and $\lambda + \lambda_k = 1$ we get that:

$$\sum_{i=1}^{k} \lambda_i a_i . = \lambda \left(\frac{1}{\lambda} \sum_{i=1}^{k-1} \lambda_i a_i \right) + \lambda_k a_k \in A.$$

Recall that a *convex hull* of a set $A \subset \mathbb{R}^d$ is the smallest inclusionwise convex set containing A, i.e.

$$conv(A) := \bigcap_{\substack{A \subset C \subset \mathbb{R}^d \\ C \text{ is convex}}} C,$$

since an intersection of convex sets is convex. A basic result gives us a way to describe convex hulls in a 'constructive' way

Lemma 3.2.2. *Given $A \subset \mathbb{R}^d$ a finite set then*

$$conv(A) = \left\{ \sum_{i=1}^{n} \lambda_i a_i \, : \, n \in \mathbb{N} \, , \, \sum_{i=1}^{n} \lambda_i = 1 \, , \, a_1, a_2 \ldots, a_m \in A, \lambda_i > 0 \right\}.$$

Proof. Denote the set on the right hand side by R. We first show that R is convex, and hence $conv(A) \subset R$ since $A \subset R$. For any $x, y \in R$ we have that

$$x = \sum_{i=1}^{n} \lambda_i a_i \quad ; \quad y = \sum_{i=1}^{m} \mu_i b_i.$$

For any $t \in [0,1]$ we have that

$$t\sum_{\iota=1}^{n}\lambda_i + (1-t)\sum_{\iota=1}^{m}\mu_i = 1,$$

so by denoting

$$c_i = \begin{cases} a_i & i \le n \\ b_{i-n} & i > n \end{cases}$$

$$\eta_i = \begin{cases} t\lambda_i & i \le n \\ (1-t)\,\mu_{i-n} & i > n \end{cases}$$

we get that

$$tx + (1-t)\,y = t\sum_{i=1}^{n}\lambda_i a_i + (1-t)\sum_{i=1}^{m}\mu_i b_i = \sum_{\iota=1}^{n+m}\eta_i c_i \in R$$

Therefore R is convex.

From Lemma 3.2.1 we get that any convex set that contains A contains R, and hence $R \subset conv\,(A)$. Therefore $R = conv\,(A)\,.$ ■

A basic result for convex sets is Radon's Theorem [Rad21].

Theorem 3.1 (Radon [Rad21])**.** *Given $m \ge d+2$ points $a_1, a_2, \ldots, a_m \in \mathbb{R}^d$, there is a partition of the points to $I \uplus J = [m]$ in such a way that:*

$$conv\left(\{a_i\}_{i\in I}\right) \cap conv\left(\{a_i\}_{i\in J}\right) \neq \emptyset.$$

Proof. Consider the points $(a_1,1), (a_2,1), \ldots, (a_m,1) \in \mathbb{R}^{d+1}$. These vectors are linearly dependent because $m > d+1$. Therefore there is a non-trivial linear combination

$$\sum_{i=1}^{m}\lambda_i\,(a_i,1) = 0. \tag{3.1}$$

Partition the set $[m]$ into two sets, $I = \{i \in [m] : \lambda_i \ge 0\}$ and $J = \{i \in [m] : \lambda_i < 0\}$. Since $\sum_{i=1}^{m}\lambda_i \cdot 1 = 0$ and the fact that there exists an $i \in [m]$ such that $\lambda_i \neq 0$, we get that $|I|, |J| \neq 0$. By manipulating (3.1) we get:

$$\sum_{i\in I}\lambda_i\,(a_i,1) = \sum_{i\in J}(-\lambda_i)\,(a_i,1)$$

Note that $\sum_{i\in I}\lambda_i = \sum_{i\in J} -\lambda_i$, and denote this sum by c. Since $\sum_{i\in I}\frac{\lambda_i}{c} = 1$ and $\frac{\lambda_i}{c} \ge 0$ we get that:

$$conv\left(\{a_i\}_{i\in I}\right) \ni \sum_{i\in I}\frac{\lambda_i}{c}a_i = \sum_{i\in J}\frac{-\lambda_i}{c}a_i \in conv\left(\{a_i\}_{i\in J}\right).$$

Which finishes the proof. ∎

Using Radon's Theorem facilitates a short proof for the following classical result of Helly [Hel23].

Theorem 3.2 (Helly). *Let $K_1, K_2, \dots, K_m$ be convex sets in $\mathbb{R}^d$. If $\bigcap_{i\in I} K_i \neq \emptyset$ for any $I \subset [m]$ of size $|I| \leq d+1$, then $\bigcap_{i\in[m]} K_i \neq \emptyset$.*

Proof. We argue by induction on m. For $m \leq d+1$ the result holds by assumption. Assume that $m \geq d+2$. By induction we know that for any $i \in [m]$ there is a point

$$p_i \in \bigcap_{j\neq i} K_j.$$

Since $m \geq d+1$ we use Radon's Theorem and get a partition $I \uplus J = [m]$ for which there is $y \in R^d$

$$y \in conv\left(\{p_i\}_{i\in I}\right) \cap conv\left(\{p_i\}_{i\in J}\right).$$

For any $l \in [m]$, l is either in J or in I. Assume without loss of generality that $l \in J$. We get that for any $i \in I$

$$p_i \in \bigcap_{j\neq i} K_j \subset K_l.$$

Since K_l is convex, it follows that $y \in conv\left(\{p_i\}_{i\in I}\right) \subset K_l$. As l is arbitrary it follows that:

$$y \in \bigcap_{j\in[m]} K_j.$$

Which finishes the proof. ∎

A massive amount of work has been done on extensions and generalizations of Helly's Theorem in various directions. But before we discuss further generalizations of Helly's Theorem, we present a classical application:

Theorem 3.3 (Kirchberger [Kir03]). *Let A, B be finite sets in $\mathbb{R}^d$, such that for any $A_0 \subset A$, $B_0 \subset B$ of size $|A_0| + |B_0| \leq d+2$, there exists a hyperplane that separates A_0 from B_0. Then, A, B can be separated by a hyperplane.*

Proof. For every $a \in A$, and $b \in B$ denote

$$K_a = \left\{(u,\alpha) \in \mathbb{R}^d \times \mathbb{R} : a \cdot u > \alpha\right\} = \left\{(u,\alpha) \in \mathbb{R}^{d+1} : (u,\alpha)(a,-1) > 0\right\},$$

and

$$L_b = \left\{(v,\beta) \in \mathbb{R}^{d+1} : b \cdot v < \beta\right\} = \left\{(v,\beta) \in \mathbb{R}^{d+1} : (v,\beta)(b,-1) < 0\right\}.$$

Now, since for any $A_0 \subset A$, $B_0 \subset B$ of size $|A_0|+|B_0| \leq d+2$, there exists a hyperplane that separates them or in other words, there exists $u \in \mathbb{R}^d$ and $\alpha \in \mathbb{R}$ for which

$$\begin{array}{c} \forall a \in A_0 \,,\, u \cdot a > \alpha \\ \forall b \in B_0 \,,\, u \cdot b < \alpha \end{array} ,$$

which gives us that

$$(u, \alpha) \in \bigcap_{a \in A_0} K_a \cap \bigcap_{b \in B_0} L_b \neq \emptyset.$$

Therefore we get that for any $d+2$ sets from the family $\{K_a\}_{a \in A} \cup \{L_b\}_{b \in B}$, of convex sets in $\mathbb{R}^{d+1}$, there is an intersection thus from Helly we get

$$(u, \alpha) \in \bigcap_{a \in A} K_a \cap \bigcap_{b \in B} L_b \neq \emptyset.$$

This gives us a hyper-plane $\left\{ v \in \mathbb{R}^d : \text{u}\cdot\text{v}{=}\alpha \right\}$ for which

$$\begin{array}{c} \forall a \in A \,,\, u \cdot a > \alpha \\ \forall b \in B \,,\, u \cdot b < \alpha \end{array} .$$

Which finishes the proof. ■

The Helly number $h := h(\mathcal{K})$ of a finite family $\mathcal{K}$ is the minimal h such that the following holds: If $\mathcal{G} \subset \mathcal{K}$ and $\cap_{K \in \mathcal{G}'} K \neq \emptyset$ for all $\mathcal{G}' \subset \mathcal{G}$ such that $|\mathcal{G}'| \leq h$ then $\cap_{K \in \mathcal{G}} K \neq \emptyset$.

A Helly type theorem for a family $\mathcal{K}$ gives an upper bound for the Helly number $h(\mathcal{K})$. For example, the original Helly Theorem asserts that $h(\mathcal{K}) = d+1$, for the family $\mathcal{K}$ of convex sets in $\mathbb{R}^d$. Another example is a theorem by Amenta [Ame96] and Morris [Mor73].

Theorem 3.4. *Let $\mathcal{K}$ be a family of sets in $\mathbb{R}^d$ such that any intersection of a non-empty subfamily of $\mathcal{K}$ can be expressed as a non-empty union of at most k closed convex sets. Then $\mathcal{K}$ has Helly number at most $k(d+1)$.*

Amenta gives a very interesting proof for this theorem using generalized LP-type problems.

We are moving our attention back to $d-$representable complexes and nerves. From Helly's Theorem we get that $\partial\Delta_n$, for $n \geq d$, cannot appear as an induced subcomplex in a $d-$representable complex. Which gives us that any $d-$representable simplicial complex is completely determined by it $d-$skeleton.

We can also express Helly's Theorem in simplicial complex language.

Theorem 3.5. *Given $X \in \mathcal{K}^d$. If $\{v_1, v_2, \ldots, v_{d+1}\} \in X$, for any choice of $d+1$ vertices in V, then $X = \Delta_{|V|-1}$.*

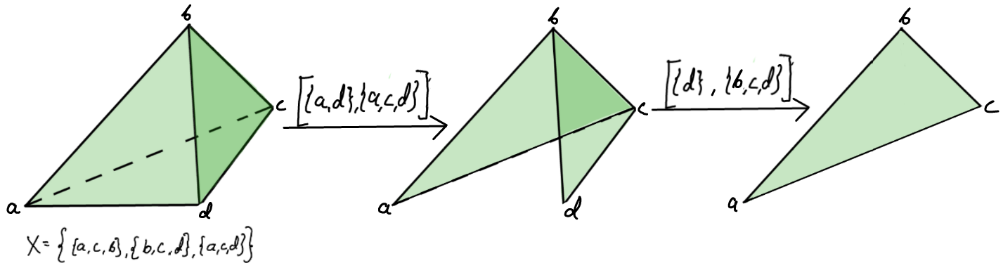


Figure 3.1: 2−collapse of a complex

In general, Helly type theorems can be formulated using nerves of the appropriate families. This sometimes gives us insight to the theorems, or helps us prove them, as we will see soon.

3.3 The $d-$Collapsibility Property

Let X be a simplicial complex with the vertex set V. For faces $\sigma \subset \tau \in X$, denote

$$[\sigma, \tau] = \{\nu \in X : \sigma \subset \nu \subset \tau\}.$$

A face $\sigma \in X$ is *free* if it is contained in a unique facet of X. An *elementary $d-$collapse* in X is the operation $X \to X - [\sigma, \tau]$ where $\sigma \in X$ is a free face contained in the unique facet $\tau \in X$, and $|\sigma| \leq d$. We will denote an elementary collapse by:

$$X \xrightarrow{[\sigma,\tau]} X - [\sigma, \tau].$$

Note that we allow a $0-$collapse, which is a collapse of the empty set. The only simplicial complexes in which a $0-$collapse is possible, are the simplices Δ_n. A *$d-$collapse* of X to a sub-simplicial complex X', denoted by $X \overset{d}{\rightsquigarrow} X'$, is a sequence :

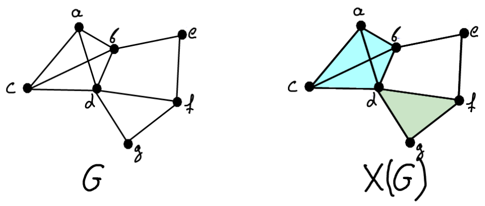
$$X = X_1 \to X_2 \to \cdots \to X_t = X',$$

where $X_i \to X_{i+1} = X - [\sigma_i, \tau_i]$ is an elementary $d-$collapse. For example, see Figure 3.1. We call X $d-$collapsible if there a exists a $d-$collapse from X to $\emptyset$. Let

$$\mathcal{C}(X) := \min\{d : X \text{ is } d\text{−collapsible}\},$$

and denote the family of all $d-$collapsible complexes by $\mathcal{C}^d$.

We first focus on $d = 1$. One can notice that a $1-$collapse of a simplicial complex really looks like an elimination order of a chordal graph. The 'only' difference is that one of them is a graph and the other is a simplicial complex. Given a graph $G = (V, E)$



[b]

Figure 3.2: Flag complex

we call its *flag* (or *clique complex*) the simplicial complex:

$$X(G) := \{f \in X : f \text{ is a clique in } G\},$$

with the vertex set V. For example, in Figure 3.2 the facets of $X(G)$ are the sets

$$\{\{a, b, c, d\}\, \{b, e\}\, \{e, f\}\, \{d, f, g\}\},$$

which are maximal cliques in G. Now since we have a way to get a simplicial complex from a graph, we would like to show that:

Proposition 3.3.1. *A simplicial complex X is $1-$collapsible if and only if there exists a chordal graph G such that $X \cong X(G)$*

To prove this proposition we need two lemmas.

Lemma 3.3.2. *For any simplicial complex X and for any $A \subset V$ we have*

$$\mathcal{C}(X[A]) \leq \mathcal{C}(X).$$

Proof. We show this in Chapter 4, see Corollary 4.1. ■

Lemma 3.3.3. *Any $1-$collapsible simplicial complex is isomorphic to some flag complex.*

Proof. Let X be $1-$collapsible simplicial complex with the vertex set V. Assume for contradiction that X is not a flag complex. Therefore there exists a set $A \subset V$ of size $|A| > 2$ for which $\binom{A}{2} \subset X$ but $A \notin X$. Pick the smallest set A for which the previous holds. Looking at the induced complex on the set A, we get $X[A] = \partial\Delta_{|A|-1}$. But then any $v \in X[A]$ is not free and thus $X[A]$ is not $1-$collapsible, which contradicts Lemma 3.3.2. Thus we get that if X is $1-$collapsible, it is isomorphic to some flag complex. ■

And now we are ready to prove Proposition 3.3.1:

Proof of Proposition 3.3.1. Lets first assume that G is chordal and show that $X(G)$ is $1-$collapsible. We show this by induction on the size of its vertex set. For $|V| = 1$, it is immediate.

Let $|V| = n$. G is chordal and thus has a simplicial vertex v_1. Therefore $N(v)$ is a maximal clique, and thus $N(v) \cup v$ is a facet in $X(G)$. Since these are exactly all the neighbors of v, $N(v) \cup v$ is the only facet containing v. Denote $G' := G[V - \{v_1\}]$ which is chordal by Claim 2.2.1. Note that $X \xrightarrow{[\{v\}, N(v) \cup \{v\}]} X[V - \{v_1\}] = X(G')$ is an elementary $1-$collapse. We are left to show that $X(G')$ is $1-$collapsible. But since the vertex set of $X(G')$ is of size $n-1$, we get by induction that $X(G')$ is $1-$collapsible and thus X is $1-$collapsible.

Now we pick a $1-$collapsible simplicial complex X, and show that it is isomorphic to a flag of a chordal graph. First, from the previous lemma we know that there is a graph $G = (V, E)$ such that $X(G) \cong X$. All that is left to show is that G is chordal. Assume for contradiction that G is not chordal. Hence there is a cycle $C \subset G$ on the vertices $V_C \subset V$ such that $|V_C| > 3$ with no chord. Since C has no chords and it is longer than 3 we get that$(X(G))[C] = C$. But then there is no free vertex in $(X(G))[C]$ and thus it is not $1-$collapsible, which is a contradiction to Lemma 4.1.1. Therefore G is chordal. ■

As mentioned before, convex sets in $\mathbb{R}$ are just intervals. That would suggest that any $1-$representable simplicial complex is isomorphic to a flag complex of some interval graph. This is true because for any $X \in \mathcal{K}^1$ there exists a family $\mathcal{F}$ of intervals such that $X \cong X(\mathcal{F})$, and $\sigma \in X(\mathcal{F})$ if and only if σ is a clique in $G(\mathcal{F})$. Recall that in Chapter 2 we have shown that any interval graph is chordal Lemma 2.2.5. Combining this with the previous lemma we get that any $1-$representable simplicial complex is $1-$collapsible.

Due to a seminal work of G. Wegner [Weg75], in which he also defines $d-$collapsibility and $d-$representability, the previous proposition is also true for higher dimensions, but we will need to work a bit harder to show it.

Theorem 3.6 (Wegner [Weg75])**.** *Let $\mathcal{K}$ be a finite family of convex sets in $\mathbb{R}^d$. Then $N(\mathcal{K})$ is $d-$collapsible.*

Remark. The following proof relies heavily on a proof from an upcoming book by Meshulam and Kozlov.

The first preliminary step before the proof of Wegner's Theorem, will be to recall a couple of definitions and prove two propositions. Recall that a *hyperplane* is a set of the form $H_{u,x} = \{v : v \cdot u = x\}$ and that $H^-_{u,x} = \{v : v \cdot u \leq x\}$ is a *half-spase*. A *convex polytope*, or simply a *polytope*, in $\mathbb{R}^d$ is a convex hull of a finite set of point.

Proposition 3.3.4. *Assume we are given a family $\mathcal{K} = \{K_1, \ldots, K_n\}$ of convex sets in $\mathbb{R}^d$. Then there exist a family $\mathcal{K}' = \{K'_1, \ldots, K'_n\}$ of polytopes, such that $K'_i \subseteq K_i$, for all $1 \leq i \leq n$, and $N(\mathcal{K}) = N(\mathcal{K}')$.*

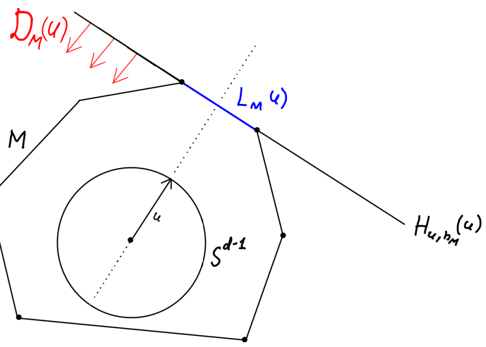


Figure 3.3: Looking at M in direction u

Proof. Let

$$\mathcal{A} = \{A \subset [n] : \cap_{i \in A} K_i \neq \emptyset\},$$

and for any $A \in \mathcal{A}$ pick a point $p_A \in \bigcap_{i \in A} K_i$. Now let the desired family of polytopes be $\mathcal{K}' = \{K_i'\}_{i=1}^n$, where

$$K_i' = conv\{p_A : i \in A \in \mathcal{A}\}.$$

We are left to show that $N(\mathcal{K}) = N(\mathcal{K}')$. On the one hand, since $K_i' \subset K_i$ for any $i \in [n]$ we get $N(\mathcal{K}') \subset N(\mathcal{K})$. On the other hand, any face $\sigma \in N(\mathcal{K})$ gives us that $\bigcap_{i \in \sigma} K_i \neq \emptyset$. Hence $\sigma \in \mathcal{A}$ and $p_\sigma \in \bigcap_{i \in \sigma} K_i'$, which gives us that $\sigma \in N(\mathcal{K})$. ∎

Let M be an arbitrary polytope in $\mathbb{R}^d$. For each unit vector $u \in S^{d-1}$, set

$$h_M(u) := \max_{v \in M} (v \cdot u). \tag{3.1}$$

Since the length of u is 1, what happens geometrically is that the polytope M is projected orthogonally on the directed line spanned by the vector u, and $h_M(u)$ is the maximum of this projection. Intuitively, we can think of $h_M(u)$ as the maximum which the set M achieves 'in the direction u'. It is well-defined because M is compact.

By shifting $H_{u,0}^-$ by $h_M(u)\,u$ we obtain the supporting half-space of M in the direction of u:

$$D_M(u) := h_M(u)\,u + H_{u,0}^- = \{v : v \cdot u \leq h_M(u)\}.$$

By definition of $h_M(u)$ we certainly have the inclusion $D_M(u) \supset M$. Let us now consider the intersection of the set M with the corresponding supporting hyperplane, and set

$$L_M(u) := M \cap \partial D_M(u) = \{v \in M : v \cdot u = h_M(u).\}$$

Since M is compact, the maximum in (3.1) is achieved, so the set $L_M(u)$ cannot be empty. In fact, it is well-known that this intersection is a polytope of lower dimension. Generically, we expect this set to consist of a single point, namely one of the vertices of the polytope M. Of course, this is not always the case, see Figure 3.3 , and we set

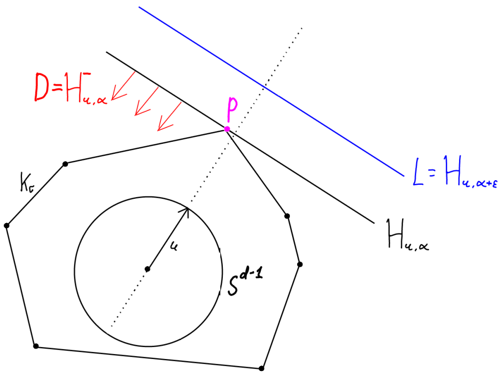


Figure 3.4: The minimal polytope K_σ and related geometric data

$$S_M := \left\{u : |L_M(u)| \geq 2 \subset S^{d-1}, \right\}$$

which we think of as a subset of singularities.

Lemma 3.3.5. *The set S_M has measure zero in S^{d-1}.*

Proof. This is immediate if one looks at the dual fan of M. Indeed, S_M is the intersection of the codimension 1 skeleton of the dual fan of M with the unit sphere. Clearly, this is a measure 0 set. ∎

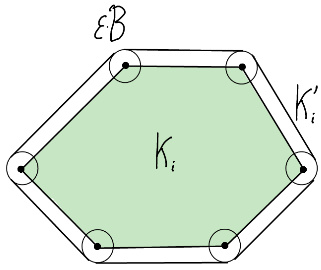
Proof of Wegner's Theorem 3.6. Let $\mathcal{K} = \{K_i\}_{i\in I}$ be a finite family of polytopes in $\mathbb{R}$. As above, for arbitrary $\sigma \subseteq I$, set $K_\sigma := \cap_{i\in\sigma} K_i$, and furthermore set

$$\widehat{\mathcal{K}} := \{K_\sigma : \sigma \in N(\mathcal{K})\}.$$

By Lemma 3.3.5 each S_{K_σ} has measure 0. The family $\mathcal{K}$ is assumed to be finite, so also the union $\cup_{\sigma\in N(\mathcal{K})} S_{K_\sigma}$ has measure 0. In particular, it cannot be the entire sphere S^{d-1}. Thus, there exists a direction $u \in S^{d-1}$ such that $|L_{K_\sigma}| = 1$ simultaneously for all $\sigma \in N(\mathcal{K})$. Set $\alpha := \min_{\sigma\in N(\mathcal{K})} h_{K_\sigma}(u)$ and set $D := H^-_{u,\alpha}$. Let σ be a minimal simplex such that $h_{K_\sigma}(u) = \alpha$ and let $p \in K_\sigma$ be the unique point such that $p \cdot u = \alpha$, see Figure 3.4 .The rest of the proof is broken into 4 steps.

Step 1. We have $|\sigma| \leq d$.

Proof of Step 1. Consider an arbitrary simplex σ', such that $\sigma' \subset \sigma$. The minimality of σ implies that $K_{\sigma'} \not\subseteq D$. Hence there exists $\varepsilon_{\sigma'} > 0$ such that the intersection $H_{u,\alpha+\varepsilon_{\sigma'}} \cap K_{\sigma'}$ is not empty. Taking the minimum over all $\sigma' \subset \sigma$ we find a constant $\varepsilon > 0$, which only depends on σ, such that the intersection $H_{u,\alpha+\varepsilon} \cap K_{\sigma'}$ is not empty for all $\sigma' \subset \sigma$.



[H]

Figure 3.5: The fattening $K' = K_i + \epsilon B$

Suppose now that $|\sigma| \geq d+1 \geq 2$. Set $L := H_{u,\alpha+\varepsilon}$, $L_i := K_i \cap L$, for $i \in \sigma$, and consider the family $\mathcal{L} = \{L_i\}_{i\in\sigma}$ of convex sets in L. For any $\sigma' \subseteq \sigma$ we have

$$L_{\sigma'} = \cap_{i\in\sigma'} L_i = \cap_{i\in\sigma'}(K_i \cap L) = (\cap_{i\in\sigma'} K_i) \cap L = K_{\sigma'} \cap L.$$

In particular, $L_{\sigma'} \neq \emptyset$ when $\sigma' \subset \sigma$. Since $|\sigma| \geq d+1$ we can apply Helly's Theorem to the family $\mathcal{L}$ to conclude that $L_\sigma = L \cap K_\sigma \neq \varnothing$. This clearly contradicts the fact that $K_\sigma \subseteq D$ and so our assumption that $|\sigma| \geq d+1$ is dismissed. This proves Step 1. ■

Step 2. The simplex σ is contained in a unique maximal simplex $\tau \in N(\mathcal{K})$.

Proof of Step 2. Set $\tau := \{i \in [n] : p \in K_i\}$. Since $p \in \cap_{i\in\tau} K_i$, we have $\cap_{i\in\tau} K_i \neq \emptyset$, hence $\tau \in N(\mathcal{K})$. Furthermore, for every $j \in \sigma$ we have $p \in K_\sigma \subseteq K_j$, hence $j \in \tau$, and so $\sigma \subseteq \tau$.

Take an arbitrary $\eta \in N(\mathcal{K})$ such that $\sigma \subseteq \eta$. We claim that $p \in K_\eta$. Indeed, otherwise we would have $K_\eta \subset K_\sigma - \{p\} \subset \operatorname{int} D$ and therefore $h_{K_\eta}(u) < h_{K_\sigma}(u)$. This is impossible since σ was chosen so that $h_{K_\sigma}(u)$ is minimal. We conclude that $p \in K_\eta$ and therefore $\eta \subseteq \tau$. Since η was chosen arbitrarily, this proves Step 2. ■

Clearly, the steps 1 and 2 imply that removing $[\sigma, \tau]$ from $N(\mathcal{K})$ constitutes a d-collapse.

Let $\varepsilon > 0$, let B denote the closed unit ball $\overline{B}(0,1)$, and consider the family $\mathcal{K}' = \{K_1', \ldots, K_n'\}$ defined by

$$K_i' = \begin{cases} K_i & i \in \sigma, \\ K_i + \varepsilon B & i \notin \sigma. \end{cases}$$

When $i \notin \sigma$, the convex set K_i' can be thought of as an "ϵ-fattening" of K_i, see Figure 3.5 . It is obvious, and left without a proof, that for a sufficiently small $\epsilon > 0$ we have $N(\mathcal{K}') = N(\mathcal{K})$.

Step 3. For each simplex $\eta \in N(\mathcal{K})$ such that $\sigma \not\subset \eta$, we have $K_\eta' \not\subset D$

Proof of Step 3. Assume $\eta \in N(\mathcal{K})$, $\sigma \nsubseteq \eta$, and set $\sigma' := \eta \cap \sigma$, $\sigma'' := \eta \backslash \sigma$. We have $\eta = \sigma' \cup \sigma''$, $\sigma' \subset \sigma$ and $\sigma'' \cap \sigma = \emptyset$. We may have $\sigma' = \emptyset$, but we definitely have $\sigma'' \neq \emptyset$. Take the point $q \in K_\eta$ satisfying $u \cdot q = h_{K_\eta}(u)$. If $u \cdot q > \alpha$, then $q \notin D$ and we are done. Otherwise, by our assumption $h_{K_\eta}(u) \geq h_{K_\sigma}(u) = \alpha$, so we must have

$u \cdot q = \alpha$, i.e., $q \in H_{u,\alpha}$, see Figure 3.6 .Set $Z := (q + \epsilon B) \backslash D$, which is a half of the ϵ-ball around q. Obviously $q \in K_{\sigma''}$ implies $Z \subseteq K'_{\sigma''}$. If $\sigma' = \emptyset$, then $K'_{\sigma''} = K'_\eta$, and again we are done showing that $K'_\eta \not\subset D$. Assume finally, $\sigma' \neq \emptyset$. The simplex σ was chosen to be minimal among those with the extreme point on $H_{u,\alpha}$. Since $\sigma' \subset \sigma$, there exists a point $q' \in K_{\sigma'} - D$. Consider the interval consisting of the points $q_t = (1-t)q + tq'$, for $0 \leq t \leq 1$. The endpoints q and q' are in $K_{\sigma'}$, hence the entire interval lies in $K_{\sigma'}$. This interval intersects the half-ball above. Formally, take $0 < t < \epsilon/|q - q'|$. Then

$$|q_t - q| = |-tq + tq'| = t|q - q'| < \epsilon,$$

and $q_t \in q + \epsilon B$. On the other hand, clearly $q_t \notin D$, so $q_t \in Z$. We conclude that $q_t \in K'_{\sigma'} \bigcap K'_{\sigma''} = K'_\eta$. Thus $K'_\eta \not\subset D$. ■

Step 4. There exists a family of polytopes $\tilde{\mathcal{K}} = \{\tilde{K}_1, \ldots, \tilde{K}_n\}$, such that $N(\tilde{\mathcal{K}}) = N(\mathcal{K}) - [\sigma, \tau]$.

Proof of Step 4. Step 3 immediately implies that for each $\eta \in N(K)$, such that $\eta \notin [\sigma, \tau]$, there exists a closed half-space D'_η such that $D \cap D'_\eta = \emptyset$ and $D'_\eta \cap K'_\eta \neq \emptyset$. Set $D' := \cup_{\eta \notin [\sigma,\tau]} D'_\eta$. The set D' is itself a closed half-space since the union is taken over finitely many elements, and furthermore $D \cap D' = \emptyset$. Let us consider the family $\mathcal{K}'' = \{K''_1, \ldots, K''_n\}$, where we set

$$K''_i := K'_i \cap D', \text{ for all } 1 \leq i \leq n.$$

For all $A \subseteq [n]$, we have

$$K''_A = \cap_{i \in A} K''_i = \cap_{i \in A}(K'_i \cap D') = (\cap_{i \in A} K'_i) \cap D' = K'_A \cap D'.$$

In particular, for $\eta \in [\sigma, \tau]$ we have $K''_\eta = K'_\eta \cap D' = \emptyset$, since $K'_\eta \subseteq K'_\sigma \subset D$ and $D \cap D' = \emptyset$. On the other hand, if $\eta \notin [\sigma, \tau]$, then $K'_\eta \neq \emptyset$ if and only if $K''_\eta = K_\eta \cap D' \neq \emptyset$. We can therefore conclude that

$$N(\mathcal{K}'') = N(\mathcal{K}) - [\sigma, \tau].$$

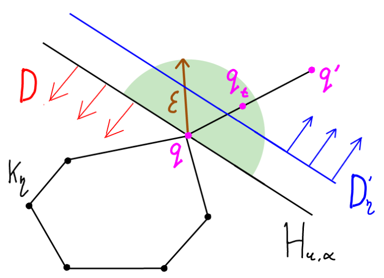


Figure 3.6: Locating the point q_t

As a last step we use Proposition 3.3.4 to replace the family $\mathcal{K}''$ with a family of polytopes $\tilde{\mathcal{K}}$, such that $N(\mathcal{K}'') = N(\tilde{\mathcal{K}})$. This proves Step 4. ∎

It is clear how Wegner's Theorem now follows from these four steps. We simply perform the d-collapse $[\sigma, \tau]$ on the nerve complex $N(\mathcal{K})$ and at the same time replace the family $\mathcal{K}$ with the family $\tilde{\mathcal{K}}$. Eventually, we end up with a complete d-collapsing sequence of the original nerve complex. ∎

Remark. Note that the inclusion $\mathcal{K}^d \subsetneq \mathcal{C}^d$ is strict. Tancer and Matousek showed even more than $\mathcal{K}^d \subsetneq \mathcal{C}^d$ in [MT09]. They show that for any $d \in \mathbb{N}$ there exists a simplicial complex X with $\mathcal{C}(X) = d$ but $\mathcal{K}(X) = 2d - 1$.

The following result relies strongly on Wegner's Theorem, and gives us a bit more information about $d-$repeatability.

Lemma 3.3.6 ([Eck85]). *Given a simplicial complex $X \in \mathcal{K}^d$ and $v \in V$ for which $St(v, X) \neq X$. There exists a $d-$*collapse $X \xrightarrow{[\sigma,\tau]} X_\sigma$ such that $St(v, X) \in X_\sigma$ and $X_\sigma \in \mathcal{K}^d$

Proof. Since $X \in \mathcal{K}^d$, there is finite family of polytopes $\mathcal{K}$ in $\mathbb{R}^d$ for which $X \cong N(\mathcal{K})$. Using the proof for Wegner's Theorem, it is enough to show that there exists $u \in S^{d-1}$ for which $|L_{\mathcal{K}_\sigma}| = 1$ for any $\sigma \in N(\mathcal{K})$ and that for $\alpha := \min_{\sigma \in N(\mathcal{K})} h_{K_\sigma}$ we get that $\mathcal{K}_v \cap H^-_{u,\alpha} = \emptyset$. Pick $\sigma \notin St(v, N(\mathcal{K}))$ and denote:

$$U = \left\{ u \in S^{d-1} : \exists \alpha \in \mathbb{R}, H^-_{u,\alpha} \cap \mathcal{K}_v = \emptyset \,\&\, \mathcal{K}_\sigma \subset H^-_{u,\alpha} \right\}.$$

Since $\mathcal{K}_\sigma, \mathcal{K}_v$ are convex, compact, and $\mathcal{K}_\sigma \cap \mathcal{K}_v = \emptyset$ we get that U is of positive measure. Now since $\bigcup_{\sigma \in \mathbb{N}(\mathcal{K})} S_{\mathcal{K}_\sigma}$ is of measure 0, there is $u \in U \setminus \bigcup_{\sigma \in \mathbb{N}(\mathcal{K})} S_{\mathcal{K}_\sigma}$. And since $\alpha \leq h_{K_\sigma} < h_{K_v}$ we get $\mathcal{K}_v \cap H^-_{u,\alpha} = \emptyset$ which is exactly what we were looking for. ∎

An example for an application of $d-$collapsibility is a nice proof for the colorful Helly's Theorem. Originally shown by Lovász [Bar82] (but appears in a paper by Bárány).

Theorem 3.7 (Colorful Helly). *Let $\mathcal{K}_1, \mathcal{K}_2, \ldots, \mathcal{K}_{d+1}$ finite families of convex sets in R^d. If $\bigcap_{i=1}^{d+1} K_i \neq \emptyset$ for all choices of $K_1 \in \mathcal{K}_1, K_2 \in \mathcal{K}_2, \ldots, K_{d+1} \in \mathcal{K}_{d+1}$ then $\bigcap_{F \in \mathcal{K}_i} F \neq \emptyset$ for some $1 \leq i \leq d+1$.*

Proof. Denote $X := N\left(\bigcup_{i=1}^{d+1} \mathcal{K}_i\right)$ the nerve of the union of the families. Note that for any $K_1 \in \mathcal{K}_1, K_2 \in \mathcal{K}_2, \ldots, K_{d+1} \in \mathcal{K}_{d+1}$ the face $\{K_1, K_2, \ldots, K_{d+1}\} \in X$, and that to prove the theorem it is enough to show that for some $1 \leq i \leq d+1$ $\mathcal{K}_i \in X$. From Theorem 3.6 we know that X is $d-$collapsible, thus we get a $d-$collapse

$$X = X_1 \xrightarrow{[\sigma_1,\tau_1]} X_2 \xrightarrow{[\sigma_2,\tau_2]} \ldots \xrightarrow{[\sigma_{t-1},\tau_{t-1}]} X_t = \emptyset \qquad (3.2)$$

We call the face ν colorful if $|\nu \cap \mathcal{K}_i| \leq 1$ for any $i \in [d+1]$. Since all such faces exist in X we can pick $1 \leq k \leq t-1$ such that $[\sigma_k, \tau_k]$ contains a colorful face ν of size d, but a $[\sigma_j, \tau_j]$ for any $j < k$ does not contain a colorful face of size d. Assume without loss of generality that $\nu = \{v_1, v_2, \dots v_d\}$ for $v_j \in \mathcal{K}_j$. Since ν is the first colorful face to collapse we know that the faces $\nu \cup \{u\} \in X_k$ for any $u \in V_{d+1}$. Since ν is a free face with a maximal face τ_k we get that:

$$\bigcup_{u \in V_{d+1}} \nu \cup \{u\} = \nu \cup V_{d+1} \subset \tau_k.$$

This finishes the proof since we showed that $V_{d+1} \in X_i \subset X$. ■

For other application of $d-$collapsibility and further detail one can look at [Tan11][Eck85]*[Tan09] [AK85].*

3.4 The $d-$Leray Property

A simplicial complex X is $d-$Leray, denoted by $X \in \mathcal{L}^d$, if for all induced subcomplex $Y \subset X$ we have $\tilde{H}_i(Y) = 0$ for any $i \geq d$. The Leray number of a given complex is

$$\mathcal{L}(X) = \min\{d \,:\, X \text{ is } d\text{−Leray}\}.$$

For example $\mathcal{L}(\Delta_n) = 0$ and for the complex X in the Figure 3.1 $\mathcal{L}(X) = 2$.

The first thing to notice is that an elementary $d-$collapse $X \xrightarrow{[\sigma,\tau]} Y$ does not change the homology in dimensions higher then d, meaning:

Lemma 3.4.1. *Given* $X \xrightarrow{[\sigma,\tau]} Y$ *an elementary* $d-$*collapse. Then* $H_i(X) = H_i(Y)$ *for any* $i \geq d$.

Proof. To show the lemma its enough to consider the two following cases:

1. $\sigma \neq \tau$. Then Y is a deformation retract of X hence homotopically equivalent to X. And thus $H^i(X) \cong H^i(Y)$ for all i.

2. $\sigma = \tau$. Then $Y = X - \{\sigma\}$, and as $\dim \sigma \leq d-1$ it follows that $Y(i) = X(i)$ for all $i \geq d$ and thus $H^i(X) \cong H^i(Y)$ for all $i \geq d$. ■

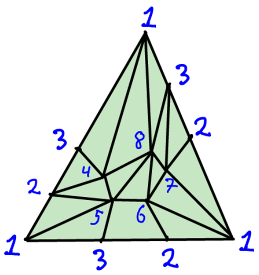
From this lemma we can conclude that:

Theorem 3.8. *Any simplicial complex which is* $d-$*collapsible is also* $d-$*Leray, i.e.* $\mathcal{C}^d \subset \mathcal{L}^d$.

Proof. Pick $X \in \mathcal{C}^d$, by Lemma 3.3.2 any induced subcomplex $Y \subset X$ has $Y \overset{d}{\rightsquigarrow} \emptyset$. From the previous lemma we can deduce that

$$H_i(Y) = H_i(\emptyset) = 0,$$

for any $i \geq d$. Which exactly means that $X \in \mathcal{L}^d$. ■



[h]
Dunce hat is a the simplicial complex we get by identify the faces denoted by the same numbers

Figure 3.1: Dunce hat

For the other side of the inclusion, there are 2 case, $d = 1$ and $d \geq 2$. First for $d = 1$:

Lemma 3.4.2. $\mathcal{L}^1 \subset \mathcal{C}^1$.

Proof. Let X be a simplicial complex in $\mathcal{L}^1$ with a vertex set V. To show that $X \in \mathcal{C}^1$ is enough to show that it is isomorphic to a flag complex of some chordal graph.

First assume for contradiction that X is not a flag complex. Like in the proof of Lemma 3.3.3 we can find a subset $A \subset V$ such that $X[A] = \partial\Delta_n$ for $n \geq 2$. Therefore $H_n(X[A]) \neq 0$ and hence $\mathcal{L}(X) > 1$ which is a contradiction to our assumption. Therefore we have graph $G = (V, E)$ such that $X(G) \cong X$. Assume for contradiction that G is not chordal. But then similarly to the second part of Proposition 3.3.1 we can find a vertex set V_C such that $(X(G))[V_C] = C$ where C is a cycle. Therefore $H_1((X(G))[V_C]) \neq 0$ which gives us that $X \notin \mathcal{L}^1$ which is a contradiction to our assumption. ∎

However for $d \geq 2$ there is a strict inclusion $\mathcal{C}^d \subsetneq \mathcal{L}^d$. One such example is the Dunce hat complex, see Figure 3.1 . For any induced subcomplex Y we have that $\tilde{H}_i(Y) = \emptyset$ for any $i \geq 2$, but for example $\tilde{H}_1(X[\{1,2,3\}]) \neq \emptyset$, therefore $X \in \mathcal{L}^2$. On the other hand any 1 dimensional face $\sigma \in X$ is not free and therefore $X \notin \mathcal{C}^2$. In general using this example and a lemma about the join of simplicial complex the authors of [MT09] constructed for any $d > 1$ a simplicial complex for which $\mathcal{C}(X) = 3d$ but $\mathcal{L}(X) = 2d$. Summing the difference between $d-$representability, $d-$collapsibility and $d-$Leray we get for $d \geq 2$

$$\mathcal{K}^d \subsetneq \mathcal{C}^d \subsetneq \mathcal{L}^d.$$

Chapter 4

Combinatorics of $d-$Collapsible complexes

4.1 Preliminaries

First we recall the notion of $d-$collapsibility : Given a simplicial complex X an *elementary $d-$collapse* is the operation $X \to X - [\sigma, \tau]$, where $\sigma \in X$ is a free face contained in the unique facet $\tau \in X$, $|\sigma| \leq d$, and $[\sigma, \tau]$ is a set interval. A *$d-$collapse* of X to a simplicial subcomplex X', denoted by $X \overset{d}{\rightsquigarrow} X'$, is a sequence of elementary $d-$collapses

$$X = X_1 \to X_2 \to \dots X_{t-1} \to X_t = X',$$

where $X_i \to X_{i+1} = X_i - [\sigma_i, \tau_i]$ is an elementary $d-$collapse. Denote by:

$$\mathcal{C}(X) := \min \{d : X \text{ is } d\text{−collepsible to } \emptyset\}.$$

We call X $d-$collapsible if $\mathcal{C}(X) = d$. Recall that for $A \subset V$ we denote by $X[A]$ the induced subcomplex on the vertices A.

Lemma 4.1.1. *Let X be a simplicial complex and $A \subset V$. If $X \overset{d}{\rightsquigarrow} Y$ then $X[A] \overset{d}{\rightsquigarrow} Y[A]$.*

Proof. Given an elementary d-collapse $X \xrightarrow{[\sigma,\tau]} X_\sigma$, if the face $\sigma \not\subset A$ then $X_\sigma[A] = X[A]$. So, from now we assume that $\sigma \subset A$. The face σ is free in $X[A]$ and $\tau \cap A$ is the unique facet in $X[A]$ containing σ, since τ is the unique facet in X containing σ. Therefore σ is collapsible in $X[A]$. Denote $Z := X[A] - [\sigma, A \cap \tau]$. We show that $Z = X_\sigma[A]$, from which we can conclude that for any elementary $d-$collapse $X \xrightarrow{[\sigma,\tau]} X_\sigma$ there is a elementary $d-$collapse $X[A] \xrightarrow{[\sigma,\tau\cap A]} X_\sigma[A]$.

On the one hand if $\nu \in X_\sigma[A]$ then $\sigma \not\subset \nu$ and $\nu \subset A$. Thus $\nu \in X[A]$ and $\nu \notin [\sigma, A \cap \tau]$ hence $\nu \in Z$. On the other hand if $\nu \in Z$ we know that $\nu \notin [\sigma, A \cap \tau]$ and $\nu \subset A$. Thus $\nu \notin [\sigma, \tau]$ and $\nu \in X[A]$ hence $\nu \in X_\sigma[A]$. Therefore $Z = X_\sigma[A]$ which follows from the double inclusion.

Let the $d-$collapse from X to Y be:

$$X = X_1 \xrightarrow{[\sigma_1,\tau_1]} X_2 \xrightarrow{[\sigma_2,\tau_2]} \ldots X_{t-1} \xrightarrow{[\sigma_{t-1},\tau_{t-1}]} X_t = Y.$$

Using repeatedly what we showed above gives us a $d-$collapse:

$$X_1 = X[A] \xrightarrow{[\sigma_1,\tau_1\cap A]} X_2[A] \xrightarrow{[\sigma_2,\tau_2\cap A]} \ldots X_{t-1}[A] \xrightarrow{[\sigma_{t-1},\tau_{t-1}\cap A]} X_t[A] = Y[A],$$

which is a $d-$collapse of $X[A]$ to $Y[A]$. ■

An important observation, that we will use and already used in the previous chapter, is:

Corollary 4.1. *For any simplicial complex X and for any $A \subset V$ we have*

$$\mathcal{C}(X[A]) \leq \mathcal{C}(X)$$

Proof. Let $\mathcal{C}(X) = d$. Since X is $d-$collapsible we have a $d-$collapse $X \overset{d}{\rightsquigarrow} \emptyset$. By Lemma 4.1.1 we get $X[A] \overset{d}{\rightsquigarrow} \emptyset[A] = \emptyset$ and hence $\mathcal{C}(X[A]) \leq \mathcal{C}(X)$. ■

Note, that the Corollary 4.1 was first shown in [Weg75] with a proof very similar to the one for Lemma 4.1.1.

Another subcomplex that does not increase the $d-$collapsibility is the link of a face in X:

Lemma 4.1.2. *Let X be a simplicial complex. For any $f \in X$*

$$\mathcal{C}(Lk(f,X)) \leq \mathcal{C}(X). \tag{4.1}$$

Proof. Pick an elementary d-collapse $X \xrightarrow{[\sigma,\tau]} X_\sigma$. We first note that if $\sigma \subset f$ then for any two faces in $\tau_1, \tau_2 \in Lk(f,X)$, their union is also in the link, i.e. $\tau_1 \cup \tau_2 \in Lk(f,X)$. This is true since $\sigma \subset f \cup \tau_i \in X$ for $i \in [2]$, and since σ is free in X we get that

$$f \cup \tau_1 \cup \tau_2 \subset \tau \in X,$$

which gives us that $\tau_1 \cup \tau_2 \in Lk(f,X)$. But since for any two faces in $Lk(f,X)$ their union is also in $Lk(f,X)$, we get that $Lk(f,X) = \Delta_n$, where $n = |Lk(f,X)|$. Therefore $\mathcal{C}(Lk(f,X)) = \mathcal{C}(\Delta_n) = 0$.

Now, assume that $\sigma \not\subset f$. We show that there exist a $d-$collapse $Lk(f,X) \overset{d}{\rightsquigarrow} Lk(f,X_\sigma)$. First, if the face $\sigma \notin Lk(f,X)$, then $Lk(f,X) = Lk(f,X_\sigma)$. Hence taking the 'empty' collapse we get that

$$Lk(f,X) \overset{d}{\rightsquigarrow} Lk(f,X) = Lk(f,X_\sigma).$$

On the other hand if $\sigma \in Lk\,(f, X)$, we assume for contradiction that σ is not free in $Lk\,(f, X)$. Therefore there exists $\tau_1, \tau_2 \in Lk\,(f, X)$ such that $\sigma \subset \tau_1, \tau_2$, but $\tau_1 \cup \tau_2 \notin Lk\,(f, X)$. From the definition of $Lk\,(f, X)$ we know that $\tau_i \cup f \in X$ for $i \in [2]$. Since $\sigma \in \tau_i \cup f$ and σ is free in X we know that

$$f \cup \tau_1 \cup \tau_2 \subset \tau \in X$$

but then $\tau_1 \cup \tau_2 \in Lk\,(f, X)$ which is a contradiction to our assumption. Similarly we get that $\tau \in Lk\,(f, X)$, and since $Lk\,(f, X) \subset X$ it is the unique facet containing σ. And finally, the simplicial complex $Lk\,(f, X) - [\sigma, \tau]$ is exactly $Lk\,(f, X_\sigma)$.

Denote X's $d-$collapse:

$$X = X_1 \xrightarrow{[\sigma_1,\tau_1]} X_2 \xrightarrow{[\sigma_2,\tau_2]} \dots X_{t-1} \xrightarrow{[\sigma_{t-1},\tau_{t-1}]} X_t = \emptyset.$$

Let $i \in [t-1]$ be the first index such that $\sigma_i \subset f$. From what we showed above we get the $d-$collapse:

$$Lk\,(f, X) = Lk\,(f, X_1) \xrightarrow{[\sigma_1,\tau_1]} Lk\,(f, X_2) \xrightarrow{[\sigma_2,\tau_2]} \dots Lk\,(f, X_{i-1}) \xrightarrow{[\sigma_{i-1},\tau_{i-1}]} Lk\,(f, X_i)\,.$$

We also showed that since $\sigma \subset f$, $Lk\,(f, X_i) = \Delta_n$, for $n = |Lk\,(f, X_i)|$. Therefore combining both of these collapses we get a $d-$collapse

$$Lk\,(f, X) \overset{d}{\rightsquigarrow} Lk\,(f, X_i) = \Delta_n \overset{0}{\rightsquigarrow} \emptyset,$$

which gives us (4.1). ∎

Another interesting property of $d-$collapsible complexes is that one can collapse it to a subcomplex which is $(d-1)$-collapsible using only elementary $d-$collapses. To show this we will need the following:

Lemma 4.1.3. *Let $X \xrightarrow{[\sigma,\tau]} Y$ be an elementary $d-$collapse. For $k \geq d$ there is a series of elementary $k-$collapses from X to Y', where:*

$$Y' := X - \{f \in [\sigma, \tau] : |f| \geq k\}\,.$$

Proof. We argue by induction on the size $(k-d)$. For $(k-d) = 1$, let $\{v_i\}_{i=1}^n = (\tau \backslash \sigma)$. For $i \in [n]$, define $\eta_i := \sigma \cup \{v_i\}$. These are exactly all the faces of size k contained in the interval $[\sigma, \tau]$. Denote $Y_i = X - \cup_{j=1}^i [\eta_j, \tau]$. We will show for each $i \in [n]$ that $\eta_i \in Y_i$ is a free face and that collapsing it leaves us with Y_{i+1}. Namely, we would get the series of elementary $k-$collapses:

$$X = Y_1 \xrightarrow{[\eta_1,\tau_1]} Y_2 \xrightarrow{[\eta_2,\tau_2]} \dots Y_n \xrightarrow{[\eta_n,\tau_n]} Y_{n+1},$$

where $Y_{n+1} = Y'$. Assume for contradiction that there is η_i that is not free in Y_i. In this case there are $\rho, \rho' \in Y_i$ such that $\eta_i \subset \rho, \rho'$ but $\rho \cup \rho' \notin Y_i$. Since $\rho \cup \rho' \notin Y_i$ there is a $j < i$ for which $\eta_j \subset \rho \cup \rho'$. But since $\eta_j = \sigma \cup \{v_j\}$ then, without loss of generality, $v_j \subset \rho$, hence $\rho \in [\eta_i, \tau_i]$, and thus $\rho \notin Y_i$. This is a contradiction to our assumption, therefore η_i is a free face. Denote its facet by τ_i. Now, on the one hand for any $f \in Y_i - Y_{i+1}$, f contains η_i. Thus $f \in [\eta_i, \tau_i]$ and

$$(Y_i - [\eta_i, \tau_i]) \subset Y_{i+1}.$$

On the other hand since $[\eta_i, \tau_i] \subset [\eta_i, \tau]$

$$(Y_i - [\eta_i, \tau_i]) \supset Y_{i+1}.$$

This finishes the base case for our induction.

Next, assume that $(k-d) = n > 1$, by the induction hypothesis we have a series of elementary $(k-1)$ −collapses:

$$X = X_1 \xrightarrow{[\sigma_1, \tau_1]} X_2 \xrightarrow{[\sigma_2, \tau_2]} \dots X_{t-1} \xrightarrow{[\sigma_{t-1}, \tau_{t-1}]} X_t, \tag{4.2}$$

where

$$X_t = X - \{f \in [\sigma, \tau] : |f| \geq k-1\}.$$

By the base case of the induction, we can get for the collapse $X_1 \xrightarrow{[\sigma_1, \tau_1]} X_2$ a series of elementary $k-$collapses:

$$X_1 = W_1^1 \xrightarrow{[\sigma_1^1, \tau_1^1]} W_2^1 \xrightarrow{[\sigma_2^1, \tau_2^1]} \dots \xrightarrow{[\sigma_{s_1-1}^1, \tau_{s_1-1}^1]} W_{s_1}^1 =: X_2',$$

where

$$X_2' = X_1 - \{f \in [\sigma_1, \tau_1] : |f| \geq k\}.$$

Note that $X_2'(m) = X_2(m)$ for every $m \geq k$ and therefore the face σ_2 is free in X_2'. Denote

$$X_i' := X_{i-1} - \{f \in [\sigma_{i-1}, \tau_{i-1}] : |f| \geq k\}.$$

Arguing by induction we get that for each $\ell \leq t$, there is sequence of elementary $k-$collapses:

$$X_{\ell-1}' \rightsquigarrow X_\ell' := X_{\ell-1}' - \{f \in [\sigma_{\ell-1}, \tau_{\ell-1}] : |f| \geq k\}.$$

Note that for all $m \geq k$ we have $X_\ell'(m) = X_\ell(m)$, hence we get that σ_ℓ is free in X_ℓ'. Therefore we get a series of elementary $k-$collapse from X to X_t' . Finally, since

$$\bigcup_{i=1}^{t} \{f \in [\sigma_i, \tau_i] : |f| \geq k\} = \{f \in [\sigma, \tau] : |f| \geq k\},$$

$X_t' = Y'$. ∎

Using this lemma we get:

Lemma 4.1.4. *Let X be a simplicial complex, then there is a $\mathcal{C}(X)-$collapse:*

$$X \overset{\mathcal{C}(X)}{\rightsquigarrow} X^{(\mathcal{C}(X)-2)}$$

Proof. Let $\mathcal{C}(X) = d$, hence there is a $d-$collapse

$$X = X_1 \xrightarrow{[\sigma_1,\tau_1]} X_2 \xrightarrow{[\sigma_2,\tau_2]} \dots \xrightarrow{[\sigma_{t-1},\tau_{t-1}]} X_t = \emptyset$$

Let $Z_1 = X$ and denote $Z_{i+1} := Z_i - \{f \in [\sigma_i, \tau_i] : |f| \geq d\}$. Using Lemma 4.1.3 we get that there is sequence of $d-$elementary collapses $X_{i-1} \rightsquigarrow Z_i$. Since $Z_i(m) = X_i(m)$ for any $m > d$, we have that there is a sequence of $d-$elementary collapses $Z_{i-1} \rightsquigarrow Z_i$. Finally, because

$$\bigcup_{i=1}^{t-1} \{f \in [\sigma_i, \tau_i] : |f| \geq d\} = \{f \in X : |f| \geq d\}$$

we get that

$$X = Z_1 \overset{d}{\rightsquigarrow} Z_{t-1} = X^{(d-2)}$$

which finishes the proof. ■

Let X, Y be simplicial complexes with the vertex sets V_X, V_Y respectively, where $V_Y \cap V_X = \emptyset$. Recall the *join* of X and Y is the simplicial complex with the vertices $V_X \dot\cup V_Y$,

$$X * Y := \left\{\sigma_X \cup \sigma_Y \in 2^{V_X \dot\cup V_Y} : \sigma_X \in X, \sigma_Y \in Y\right\}.$$

This brings us to another lemma which will be useful later on:

Lemma 4.1.5. *Let X be a simplicial complex with the vertex set V, and $X \xrightarrow{[\sigma,\tau]} X_\sigma$ be an elementary d-collapse. Then for $v \notin V$, σ is a free face in $(X * \{v\})$, and there is an elementary $d-$collapse:*

$$X * \{v\} \xrightarrow{[\sigma,\tau\cup\{v\}]} (X * \{v\})_\sigma = X_\sigma * \{v\},$$

where $(X * \{v\})_\sigma := (X * \{v\}) - [\sigma, \tau \cup \{v\}]$.

Proof. First notice that $\tau \cup \{v\}$ is a unique facet in $X * \{v\}$ containing σ. This is true since if there was another face containing it, $\sigma \subset \tilde{\tau}$, such that $\tilde{\tau} \not\subset \tau \cup \{v\}$ then $\sigma \subset (\tilde{\tau} \backslash \{v\})$ and $(\tilde{\tau} \backslash \{v\}) \not\subset \tau$. But that would contradict the fact that σ is collapsible, and therefore σ is free.

For any face $f \in (X * \{v\})_\sigma$, we know that $\sigma \not\subset f$ hence $f \backslash \{v\} \in X_\sigma$ therefore $f \in X_\sigma * \{v\}$. On the other hand if we pick $f \in X_\sigma * \{v\}$ then $\sigma \not\subset f$ which means that $f \notin [\sigma, \tau \cup \{v\}]$ and hence $f \in (X * \{v\})_\sigma$. Therefor $(X * \{v\})_\sigma = X_\sigma * \{v\}$. ■

4.2 The collapsibility of intersections

Recall that given X and Y two simplicial complexes on vertex set V_X and V_Y respectively, their *intersection* is:

$$X \cap Y := \left\{ f \in 2^{V_X \cap V_Y} | f \in X \wedge f \in Y \right\}.$$

For any set $\tau \in X \cap Y$ and a subset $\sigma \subset \tau$ we have from monotonicity that $\sigma \in X$ and $\sigma \in Y$ thus $\sigma \in X \cap Y$. Therefore the intersection of any two simplicial complexes is also a simplicial complex with the vertices $V_X \cap V_Y$.

In this section we discuss how does $d-$collapsibility interact with intersection. More precisely, if one knows $\mathcal{C}(X)$ and $\mathcal{C}(Y)$ what can be said about $\mathcal{C}(X \cap Y)$?

First, notice that bounding $\mathcal{C}(X \cap Y)$ from below (by something that is not trivial) is impossible. Indeed for any $k, m \in \mathbb{N}$ one can take $V = [m] \dot{\cup} [n]$, $X = \partial \Delta_m$, and $Y = \partial \Delta_n$ for which we have $\mathcal{C}(X) = m$ and $\mathcal{C}(Y) = n$ but $\mathcal{C}(X \cap Y) = \mathcal{C}(\emptyset) = 0$. Fortunately, it turns out that there exists an upper bound, and the heart of its proof is the following lemma:

Lemma 4.2.1. *Let X be a simplicial complex with an elementary $k-$collapse $X \xrightarrow{[\sigma,\tau]} X_\sigma$. If the simplicial complex Y satisfies $\mathcal{C}(Y) = \ell$, then there exists a $(k+\ell)-$collapse*

$$X \cap Y \overset{k+\ell}{\rightsquigarrow} X_\sigma \cap Y.$$

Proof. First, if $\sigma \notin Y$, then $[\sigma, \tau] \not\subset Y$ and therefore $X \cap Y = X_\sigma \cap Y$. Hence, from now on we will assume that $\sigma \in Y$. Using Lemma 4.1.2 and Corollary 4.1 we have that $\mathcal{C}(Lk(\sigma, Y[\tau])) \leq \ell$. Hence there is a $\ell-$collapsing sequence:

$$Lk(\sigma, Y[\tau]) = L_1 \xrightarrow{[\nu_1,\eta_1]} L_2 \xrightarrow{[\nu_2,\eta_2]} \dots L_{t-1} \xrightarrow{[\nu_{t-1},\eta_{t-1}]} L_t = \emptyset.$$

Denote $Z_i = X \cap Y - \bigcup_{j<i} [\nu_j \cup \sigma, \eta_j \cup \sigma]$. To prove the lemma it would be enough to show that $Z_t = X_\sigma \cap Y$, and that for any $i \in [s]$ the face $\nu_i \cup \sigma$ is free in Z_i and $\eta_i \cup \sigma$ is the facet containing it. This would give us the $d-$collapse:

$$X \cap Y = Z_1 \xrightarrow{[\nu_1\cup\sigma,\eta_1\cup\sigma]} Z_2 \xrightarrow{[\nu_2\cup\sigma,\eta_2\cup\sigma]} \dots Z_{t-1} \xrightarrow{[\nu_{t-1}\cup\sigma,\eta_{t-1}\cup\sigma]} Z_t = X_\sigma \cap Y,$$

where $d = \max_{j\in[t-1]} \{|\nu_j \cup \sigma|\} \leq k + \ell$.

In order to show that $Z_t = X_\sigma \cap Y$ it is enough to show that $\bigcup_{j=1}^{t-1} [\nu_j \cup \sigma, \eta_j \cup \sigma] = [\sigma, \tau] \cap Y$. On the one hand, for any $\rho \in [\nu_i \cup \sigma, \eta_i \cup \sigma]$ we get that:

$$\sigma \subset \rho \subset \eta_i \cup \sigma \in Y[\tau].$$

Hence $\rho \in [\sigma, \tau]$, and $\rho \in Y$ for any $i \in [t-1]$ which gives us that $\rho \in [\sigma, \tau] \cap Y$. On the other hand, for any $\rho \in [\sigma, \tau] \cap Y$, look at the face $\tilde{\rho} := (\rho \backslash \sigma) \in Lk(\sigma, Y[\tau])$ and pick the minimal $i \in [t-1]$ for which $\nu_i \subset \tilde{\rho}$. From the minimality of i we have that $\tilde{\rho} \in L_i$ and because ν_i is free in L_i we get that $\tilde{\rho} \in [\nu_i, \eta_i]$. Hence $(\rho - \sigma) \cup \sigma = \rho \in [\sigma \cup \nu_i, \eta_i \cup \sigma]$ and thus $\rho \in \bigcup_{j=1}^{t-1} [\nu_j \cup \sigma, \eta_j \cup \sigma]$. Therefore we can conclude that $Z_t = X_\sigma \cap Y$

Now, note that $\eta_i \cup \sigma \in Z_i$. Indeed this is true since $\eta_i \notin [\nu_j, \eta_j]$ for any $j < i$, and therefore $\eta_i \notin [\nu_j \cup \sigma, \eta_j \cup \sigma]$. We conclude that also $\nu_i \in Z_i$ since $\nu_i \subset \eta_i$. Let $\rho \in Z_i$ for which $\sigma \cup \nu_i \subset \rho$. Assume for contradiction that $\rho \not\subset \eta_i \cup \sigma$. The face σ is free in X inside a maximal face τ, hence $\sigma \subset \rho \subset \tau$. Therefore $(\rho - \sigma) \in Lk\left(\sigma, Y[\tau]\right)$ and since ν_i is free in L_i, we get that $(\rho - \sigma) \in [\nu_j, \eta_j]$ for some $j < i$. But then we get that $(\rho - \sigma) \cup \sigma = \rho \in [\nu_j \cup \sigma, \eta_j \cup \sigma]$ which is a contradiction to the assumption that $\rho \in Z_i$. Therefore $\nu_i \cup \sigma$ is a free face in Z_i with a unique facet $\eta_i \cup \sigma$, which finishes the proof. ■

We want to point the reader's attention to the fact that the previous lemma is a good example to the basic technique of proving properties of $d-$collapsibility. One first proves "the property holds for an elementary collapse" and then just uses it repeatedly:

Theorem 4.2. *For X, Y simplicial complexes*

$$\mathcal{C}(X \cap Y) \leq \mathcal{C}(X) + \mathcal{C}(Y). \tag{4.1}$$

Proof. Let $\mathcal{C}(X) = k$ and $\mathcal{C}(Y) = \ell$. Let

$$X = X_1 \xrightarrow{[\sigma_1,\tau_1]} X_2 \xrightarrow{[\sigma_2,\tau_2]} \dots X_{t-1} \xrightarrow{[\sigma_{t-1},\tau_{t-1}]} X_t = \emptyset,$$

be a k-collapse of X. Denote $Z_i = X_i \cap Y$, and note that $Z_t = X_t \cap Y = \emptyset$. By the previous lemma we get that $Z_i \overset{k+\ell}{\rightsquigarrow} Z_{i+1}$ for any $i \in [t-1]$, and therefore we get the $(k+\ell)-$collapse:

$$X \cap Y = Z_1 \overset{k+\ell}{\rightsquigarrow} Z_2 \overset{k+\ell}{\rightsquigarrow} \dots Z_t \overset{k+\ell}{\rightsquigarrow} Z_{t+1} = \emptyset.$$

Which gives us Inequality (4.1) and finishes the proof. ■

An application of the Intersection Theorem 4.2 is a bound on the collapsibility of a join of two simplicial complexes. But first, we show that a join with a simplex doesn't change the $d-$collapsibility:

Lemma 4.2.2. *For any simplicial complex X*

$$\mathcal{C}(X * \Delta_n) = \mathcal{C}(X).$$

Proof. Let $\mathcal{C}(X) = d$, and denote its $d-$collapse:

$$X = X_1 \xrightarrow{[\sigma_1,\tau_1]} X_2 \xrightarrow{[\sigma_2,\tau_2]} \dots X_{t-1} \xrightarrow{[\sigma_{t-1},\tau_{t-1}]} X_t = \emptyset. \tag{4.2}$$

Notice that $\Delta_n \cong \{v_1\} * \{v_2\} * \cdots * \{v_{n+1}\}$ and hence

$$X * \Delta_n \cong X * \{v_1\} * \{v_2\} * \cdots * \{v_{n+1}\} .$$

Using repeatedly Lemma 4.1.5 on the collapse (4.2) we get the $d-$collapse:

$$X * \Delta_n = X_1 * \Delta_n \xrightarrow{[\sigma_1,\tau_1 \cup \Delta_n]} X_2 * \Delta_n \xrightarrow{[\sigma_2,\tau_2 \cup \Delta_n]} \dots \xrightarrow{[\sigma_{t-1},\tau_{t-1} \cup \Delta_n]} X_t * \Delta_n = \emptyset.$$

Since there is a $d-$collapse $\Delta_n * X \overset{d}{\rightsquigarrow} \emptyset$ we get that $\mathcal{C}(X * \Delta_n) \leq \mathcal{C}(X)$. On the other hand if there was a $(d-1)-$collapse of $X * \Delta_n$, since X is an induced subcomplex of $X * \Delta_n$, then by Lemma 4.1.1 we would get a $(d-1)-$collapse of X which would contradict the assumption that $\mathcal{C}(X) = d$. Therefore $\mathcal{C}(X * \Delta_n) = \mathcal{C}(X)$. ■

Using this lemma and the Intersection Theorem 4.2 we get:

Proposition 4.2.3. *Given X and Y simplicial complexes. Then*

$$\mathcal{C}(X * Y) \leq \mathcal{C}(X) + \mathcal{C}(Y) .$$

Proof. Denote by Δ_X the simplex with the vertices V_X of the complex X and Δ_Y the simplex with the vertices V_Y of the complex Y. Denote

$$\tilde{X} := X * \Delta_Y \ ; \ \tilde{Y} := Y * \Delta_X .$$

First, from the previous lemma we get that $\mathcal{C}\left(\tilde{X}\right) = \mathcal{C}(X)$ and $\mathcal{C}\left(\tilde{Y}\right) = \mathcal{C}(Y)$. Pick a set $f \in 2^{V_X \uplus V_Y}$, it is contained in $\tilde{X} \cap \tilde{Y}$ if and only if $f = f_X \uplus f_Y$ for some $f_x \in X$ and $f_Y \in Y$ which means that $\tilde{X} \cap \tilde{Y} = X * Y$. Using the Intersection Theorem 4.2 we can conclude that:

$$\mathcal{C}(X * Y) = \mathcal{C}\left(\tilde{X} \cap \tilde{Y}\right) \leq \mathcal{C}\left(\tilde{X}\right) + \mathcal{C}\left(\tilde{Y}\right) = \mathcal{C}(X) + \mathcal{C}(Y) .$$

Which finishes the proof. ■

The other side of the inequality, i.e. $\mathcal{C}(X * Y) \geq \mathcal{C}(X) + C(Y)$, was discussed in the paper [MT09]. They have shown that for any simplicial complex X, if we denote

$$\gamma_0(X) := \min\{d : X \text{ has a } d\text{−collapsible face}\}$$

then

Lemma 4.2.4 (Lemma 4.2 in [MT09])**.** *For every two simplicial complexes X, Y we have*

$$\gamma_0(X * Y) = \gamma_0(X) + \gamma_0(Y) .$$

In [MT09] the authors used this lemma to construct a family of complexes $\{X_n\}_{n=1}^{\infty}$ such that is X_n is not $(3n-1)-$collapsible but it is $2n-$Leray.

We note that the Intersection Theorem 4.2 is tight since we can build the following family of examples:

Example 4.2.5. Take the simplicial complexes $X = \partial\Delta_m$ and $Y = \partial\Delta_\ell$ and use them to construct $\tilde{X}, \tilde{Y}$ like in the proof of Lemma 4.2.2. First notice that by Lemma 4.2.2 $\mathcal{C}\left(\tilde{X}\right) = \mathcal{C}(X) = m$ and $\mathcal{C}\left(\tilde{Y}\right) = \mathcal{C}(Y) = \ell$. And from the fact that $\tilde{X} \cap \tilde{Y} = \partial\Delta_{m+\ell}$, we have

$$\mathcal{C}\left(\tilde{X} \cap \tilde{Y}\right) = \mathcal{C}\left(\partial\Delta_{m+\ell}\right) = m + \ell = \mathcal{C}\left(\tilde{X}\right) + \mathcal{C}\left(\tilde{Y}\right).$$

Recall Lemma 4.1.2 which gives us a bound on the $d-$collapsibility of the link of any face. As a conclusion of this lemma and of Lemma 4.2.2 we get that a bound on the star of a face:

Lemma 4.2.6. *Let X be a simplicial complex. For any $f \in X$*

$$\mathcal{C}\left(St\left(f, X\right)\right) \leq \mathcal{C}\left(X\right).$$

Proof. First note that $St\left(f, X\right) = Lk\left(f, X\right) * 2^f$. From Lemma 4.2.2 we conclude that:

$$\mathcal{C}\left(St\left(f, X\right)\right) = \mathcal{C}\left(Lk\left(f, X\right) * 2^f\right) = \mathcal{C}\left(Lk\left(f, X\right)\right).$$

From Lemma 4.1.2 we know that $\mathcal{C}\left(Lk\left(f, X\right)\right) \leq \mathcal{C}\left(X\right)$, and hence

$$\mathcal{C}\left(St\left(f, X\right)\right) = \mathcal{C}\left(Lk\left(f, X\right)\right) \leq \mathcal{C}\left(X\right).$$

4.3 Collapsing from the complete complex

We call a complex X with the vertices V *$d-$star-collapsible* if it has an order on the vertices $V = \{v_i\}_{i=1}^n$ for which, denoting $X_i := X\left[\{v_j\}_{j=i}^n\right]$, there is a collapse

$$X_i \overset{d}{\rightsquigarrow} St\left(v_i, X_{i-1}\right) \overset{d}{\rightsquigarrow} \emptyset.$$

Note that like $d-$collapsibility, if X is $d-$star-collapsible then so is any of its induced subcomplexes.

Lemma 4.3.1. *Let X be a $d-$star-collapsible complex with the vertices set V. Then for any $A \subset V$, $X[A]$ is also $d-$star-collapsible.*

Proof. Since X is $d-$star-collapsible there is an order on the vertices $\{v\}_{i=1}^n$ for which

$$X_i \overset{d}{\rightsquigarrow} St\left(v_i, X\right) \overset{d}{\rightsquigarrow} \emptyset,$$

where $X_i := X\left[\{v_j\}_{j=i}^n\right]$. Take the order on the vertices $A = \{v_{i_k}\}_{k=1}^{|A|}$ to be the induced order from the order on V. For any $1 \le \ell \le |A|$ we get:

$$X\left[\{v_{i_k}\}_{k=l}^{|A|}\right] = X_{i_\ell}[A] \underset{(\#)}{\overset{d}{\rightsquigarrow}} St\left(v_{i_\ell}, X_{i_l}\right)[A] = St\left(v_{i_\ell}, X\left[\{v_{i_k}\}_{k=l}^{|A|}\right]\right) \underset{(\#)}{\overset{d}{\rightsquigarrow}} \emptyset.$$

Where we know the collapses (#) exists from Lemma 4.1.1. Therefore we get that $X[A]$ is $d-$star-collapsible. ■

A nice property that $d-$star-collapsible complexes have is:

Proposition 4.3.2. *Let X be a simplicial complex with the vertices set V. If X is $d-$star-collapsible, than there is a $(d+1)$-collapse $\Delta_{|V|-1} \overset{d+1}{\rightsquigarrow} X$.*

Proof. We will prove the proposition by induction on $|V|$. For $|V| = 1$, it is clear.

Assume that $|V(X)| = n$. If $X = \Delta_{n-1}$ then we are done, so we can assume that $X \neq \Delta_{n-1}$. Let $v \in V$ be a vertex such that there exist a collapse $X \overset{d}{\rightsquigarrow} St(v, X)$ and denote $X_v := X[V - \{v\}]$. Notice that $|V(X_v)| = n-1$, and that from Lemma 4.3.1 X_v is $d-$star collapsible. Therefore by the induction hypothesis we have the $(d+1)$-collapse:

$$\Delta_{n-2} =: Y_1 \xrightarrow{[\sigma_1,\tau_1]} Y_2 \xrightarrow{[\sigma_2,\tau_2]} ...Y_{m-1} \xrightarrow{[\sigma_m,\tau_m]} Y_m = X_v.$$

Taking $\Delta_{n-2} * \{v\}\ (= \Delta_{n-1})$ and using Lemma 4.1.5 on each of the elementary collapse we get the $(d+1)$-collapse:

$$\Delta_{n-2}*\{v\} := Y_1*\{v\} \xrightarrow{[\sigma_1,\tau_1\cup\{v\}]} Y_2*\{v\} \xrightarrow{[\sigma_2,\tau_2\cup\{v\}]} ...Y_{m-1}*\{v\} \xrightarrow{[\sigma_{m-1},\tau_{m-1}\cup\{v\}]} Y_m*\{v\} = X_v*\{v\}.$$

Denote the $d-$collapse from X to $St(v, X)$ by:

$$X := X_1 \xrightarrow{[\nu_1,\eta_1]} X_2 \xrightarrow{[\nu_2,\eta_2]} \dots X_{k-1} \xrightarrow{[\nu_{k-1},\eta_{k-1}]} X_k = St(v, X). \tag{4.1}$$

Denote $\hat{\nu}_i := \nu_i \uplus \{v\}$, $\hat{\eta}_i := \eta_i \uplus \{v\}$, and $Z_i := X_v * \{v\} - \cup_{j=1}^{i} [\hat{\nu}_j, \hat{\eta}_j]$ for $i \in [k]$. First, notice that:

$$Z_k = St(v, X) \cup \left(\cup_{j=1}^{k} [\nu_j, \eta_j]\right) = X.$$

Hence it is enough to show that for any $i \in [k-1]$ that the face $\hat{\nu}_i$ is a free face in Z_i with a unique facet $\hat{\eta}_i$. Indeed, in this case we would get the $\ell-$collapse

$$\Delta_{n-1} \overset{d+1}{\rightsquigarrow} X_v * \{v\} = Z_1 \xrightarrow{[\hat{\nu}_1,\hat{\eta}_1]} Z_2 ... Z_{k-1} \xrightarrow{[\hat{\nu}_k,\hat{\eta}_k]} Z_k = X,$$

where $\ell \le \max_{i \in k} |\hat{\nu}_i| \le d+1$, which is exactly the desired collapse.

For any $i \in [k-1]$, $\hat{\eta}_i \in Z_i$ since otherwise there would exist $j < i$ for which $\hat{\eta}_i \in [\hat{\nu}_j, \hat{\eta}_j]$, but that would imply that $\nu_j \subset \eta_i$ which we know is impossible since (4.1) is a proper $(d+1)-$collapse. We can also conclude that $\hat{\nu}_i \in Z_i$ since $\hat{\nu}_i \subset \hat{\eta}_i$. Pick a face $\rho \in Z_i$ such that $\hat{\nu}_i \subset \rho$, and denote $\rho_v := \rho - \{v\}$. Assume for contradiction that

$\rho_v \not\subset \eta_i$. Since $\nu_{\rm i}$ is free in Z_i there exist $0 < j < i$ for which $\rho_v \subset [\nu_j, \eta_j]$, but then $\rho \subset [\hat{\nu}_j, \hat{\eta}_j]$ which would contradict the fact that $\rho \in Z_i$. Hence $\rho \subset \hat{\eta}_j$ and therefore $\hat{\nu}_i$ is a free face in Z_i with maximal face $\hat{\eta}_i$. ■

An application of Proposition 4.3.2 is a bound on the collapsibility of a union of two complexes. Recall that a union of two simplicial complexes X, Y is:

$$X \cup Y := \left\{ f \in 2^{V_X \cup V_Y} : f \in X \text{ or } f \in Y \right\}.$$

As for intersection we do not have a lower bound. Indeed, taking $X = \Delta_n \uplus \partial\Delta_m$ and $Y = \partial\Delta_n \uplus \Delta_m$ simplicial complexes with the vertices set $[n+1] \uplus [m+1]$. We would get that $\mathcal{C}(X) = n$, $\mathcal{C}(Y) = m$ but $\mathcal{C}(X \cup Y) = 1$.

We were not able to show an upper bound for general complexes, but we strongly believe that the following holds:

Conjecture 4.3.3. Let X, Y be simplicial complexes, then

$$\mathcal{C}(X \cup Y) \leq \mathcal{C}(X) + \mathcal{C}(Y) + 1.$$

Nevertheless, we are able to show a weaker version of the bound using $d-$star-collapsibility, but first:

Lemma 4.3.4. *Given a simplicial complexes Y which is $\ell-$star-collapsible. For any complex X and an elementary $k-$collapse $X \xrightarrow{[\sigma,\tau]} X_\sigma$ there exists a collapse*

$$Y \cup X \overset{k+\ell+1}{\rightsquigarrow} Y \cup X_\sigma.$$

Proof. First we notice that if $\sigma \notin Y$, then it is a free face with a maximal face τ, hence $Y \cup X \xrightarrow{[\sigma,\tau]} Y \cup X_\sigma$ and we are done. From Lemma 4.3.1 $Y[\tau]$ is $\ell-$star-collapsible and hence by Proposition 4.3.2 there is a $(\ell+1)-$collapse:

$$\Delta_{|\tau|-1} = Y_1 \xrightarrow{[\nu_1,\eta_1]} Y_2 \xrightarrow{[\nu_2,\eta_2]} \dots Y_{t-1} \xrightarrow{[\nu_{t-1},\eta_{t-1}]} Y_t = Y[\tau]. \tag{4.2}$$

Now, denote $Z_1 := X \cup Y$, and for $i \in [t-1]$ let

$$Z_i := \begin{cases} Z_{i-1} - [\sigma \cup \nu_{i-1}, \eta_{i-1}] & \sigma \subset \eta_i \\ Z_{i-1} & \sigma \not\subset \eta_i \end{cases}.$$

We will begin by showing that $Z_i \overset{k+\ell+1}{\rightsquigarrow} Z_{i+1}$, for any $i \in [t-1]$.

We start by noting, that for any $i \in [t-1]$ if $\sigma \not\subset \eta_i$ then $Z_i = Z_{i+1}$ and hence there is a collapse $Z_i \overset{k+\ell+1}{\rightsquigarrow} Z_{i+1}$, the empty one. So from now on we will assume that $\sigma \subset \eta_i$. The face η_i is in Z_i since otherwise there would exist an $j < i$ for which $\eta_i \in [\sigma \cup \nu_j, \eta_j]$ but that is not possible since (4.2) is a proper $(\ell+1)-$collapse. We also conclude that $\sigma \cup \nu_i \in Z_i$. For any face $\rho \in Z_i$ such that $\sigma \cup \nu_i \subset \rho$, we know that $\rho \subset \tau$ because

σ is a free face in τ and $\nu_i \notin Y$. Therefore, assume for contradiction that $\rho \not\subset \eta_i$, then since ν_i is free in Y_i we get that $\rho \in [\nu_j, \eta_j]$ for some $j < i$. But since $\sigma \subset \rho$ we have that $\rho \in [\sigma \cup \nu_j, \eta_j] \not\subset Z_i$, which is a contradiction to the assumption that $\rho \not\subset \eta_i$. Consequently, we get that $\sigma \cup \nu_i$ is free in a unique facet η_i and $Z_i \xrightarrow{[\sigma\cup\nu_i,\eta_i]} Z_{i+1}$. Since $|\sigma \cup \nu_i| \leq k + \ell + 1$ it gives us that $Z_i \overset{k+\ell+1}{\rightsquigarrow} Z_{i+1}$.

What is left to show is that $Z_t = Y \cup X_\sigma$. First, Let $\left\{\eta_{k_j}\right\}_{j=1}^{m}$ be the facets η_{k_j} that contain σ, we get that $Z_t = X \cup Y - \bigcup_{j=1}^{m} \left[\sigma \cup \nu_{k_j}, \eta_{k_j}\right]$. On the one hand for any face $f \in Y \cup X_\sigma$, f is either in Y and then $f \in Z_t$ since $\left[\sigma \cup \nu_{k_j}, \eta_{k_j}\right] \not\subset Y$, or $f \notin [\sigma, \tau]$ and then $f \notin \left[\sigma \cup \nu_{k_j}, \eta_{k_j}\right]$ for any j and therefore $f \in Z_t$. On the other hand assume that $f \in Z_t$ then:

$$f \notin \bigcup_{j=1}^{m} \left[\sigma \cup \nu_{k_j}, \eta_{k_j}\right] = [\sigma, \tau] - \{f \in [\sigma, \tau] : f \in Y\},$$

and hence $f \in Y \cup X_\sigma$.

Combining what we showed above we get a $(k + \ell + 1)$ −collapse

$$Y \cup X = Z_1 \overset{k+\ell+1}{\rightsquigarrow} Z_t = Y \cup X_\sigma.$$

Using the previous lemma we get the bound:

Theorem 4.3. *Let X, Y be simplicial complexes. If Y is $\mathcal{C}(Y)$* −star-collapsible, then

$$\mathcal{C}(X \cup Y) \leq \mathcal{C}(X) + \mathcal{C}(Y) + 1. \tag{4.3}$$

Proof. Let $\mathcal{C}(X) = k$ and $\mathcal{C}(Y) = \ell$. Let

$$X = X_1 \xrightarrow{[\sigma_1,\tau_1]} X_2 \xrightarrow{[\sigma_2,\tau_2]} \dots X_{t-1} \xrightarrow{[\sigma_{t-1},\tau_{t-1}]} X_t = \emptyset,$$

be a k-collapse of X. Denote $Z_i = X_i \cup Y$. Notice that $Z_t = X_t \cup Y = Y$, therefore, if we show that for every $i < t$ there exists a $(k + \ell + 1)$-collapse $Z_i \overset{k+\ell+1}{\rightsquigarrow} Z_{i+1}$, we would get a $(k + \ell + 1)$-collapse $X \cup Y \overset{k+\ell+1}{\rightsquigarrow} Y$. Using Lemma 4.3.4 that is exactly what we get. ■

Note that Theorem 4.3 is tight since we have the following family of examples:

Example 4.3.5. Take the simplicial complex $\tilde{X}$ and $\tilde{Y}$ from Example 4.2.5. First note that $\tilde{X} \cup \tilde{Y} = \partial(\Delta_{m+\ell+1})$ and $\hat{Y}$ is ℓ−star-collapsible. We get

$$\mathcal{C}\left(\tilde{X} \cup \tilde{Y}\right) = \mathcal{C}\left(\partial\left(\Delta_{m+\ell+1}\right)\right) = m + \ell + 1 = \mathcal{C}\left(\tilde{X}\right) + \mathcal{C}\left(\tilde{Y}\right) + 1.$$

To show Conjecture 4.3.3 it would be enough to show:

Conjecture 4.3.6. Let X be a simplicial complex. There is vertex $v \in V$ for which $X \overset{\mathcal{C}(X)}{\rightsquigarrow} St(v, X)$.

Indeed, using Lemma 4.1.1 and Conjecture 4.3.6 we would get that any $d-$collapsible simplicial complex X is also $d-$star-collapsible, and hence using Theorem 4.3 we can deduce Conjecture 4.3.3.

In the following section we show that there exists an interesting subclass of $d-$collapsible complexes which are $d-$star-collapsible. The class is a sort of 'strengthening' of $d-$collapsibility, which was given by Eckhoff in [Eck85].

4.4 Strongly $d-$Collapsible

In the paper [Eck85] Eckhoff gives the following definition. We call a simplicial complex X on vertexes V *strongly $d-$collapsible* if for every vertex $v \in V$ there is a $d-$collapse

$$X \overset{d}{\rightsquigarrow} St\left(v, X\right) \overset{d}{\rightsquigarrow} \emptyset.$$

Strong collapsibility plays an important role in Echoff's proof of the M. Katchalski and M. A. Perles upper bound conjecture on convex families in $\mathbb{R}^d$.

First note that strong $d-$collapsibility implies $d-$collapsibility. Eckhoff showed the following:

Theorem 4.4 (Eckhoff[Eck85])**.** *Every $d-$representable simplicial complex is strongly $d-$collapsible.*

Proof. Let $X \in \mathcal{K}^d$, and note that if $St\left(v, X\right) = X$ for all vertices $v \in V$ then we are done. Hence we assume that there is $v \in V$ for which $St\left(v, X\right) \neq X$. From Lemma 3.3.6 there is a $d-$collapse $X \xrightarrow{[\sigma,\tau]} X_\sigma$ such that $St\left(v, X\right) \subset X_\sigma$ and $X_\sigma \in \mathcal{K}^d$. Therefore as long as $St\left(v, X\right) \neq X_\sigma$ we can use this argument repeatedly until we are left with:

$$X \overset{d}{\rightsquigarrow} St\left(v, X\right),$$

and by Lemma 4.2.6 $St\left(v, X\right) \overset{d}{\rightsquigarrow} \emptyset$. ■

On the other hand not every strongly $d-$collapsible is $d-$representable. For example, the simplicial complex in Figure 2.2 is strongly $1-$collapsible but not $1-$representable. Strongly $d-$collapsible simplicial complexes interest us since:

Proposition 4.4.1. *Every simplicial complex which is strongly $d-$collapsible is $d-$star-collapsible*

Proof. Pick an order on $V = \{v_i\}_{i=1}^n$ the vertices of X, any order will work. We will show that with this sequence, we have $d-$star-collapsibility. Since X is strongly $d-$collapsible for any $i \in [n]$ there is the $d-$collapse

$$X \overset{d}{\rightsquigarrow} St\left(v_i, X\right) \overset{d}{\rightsquigarrow} \emptyset.$$

From the Lemma 4.1.1 we get:

$$X\left[\{v_j\}_{j=i}^{n}\right] \overset{d}{\rightsquigarrow} St\left(v_i, X\right)\left[\{v_j\}_{j=i}^{n}\right] = St\left(v_i, X\left[\{v_j\}_{j=i}^{n}\right]\right) \overset{d}{\rightsquigarrow} \emptyset.$$

Therefore X is $d-$star-collapsible. ■

The previous proposition plus Theorem 4.4 implies that every $d-$representable simplicial complex is $d-$star-collapsible. However not every $d-$star-collapsible is strongly $d-$collapsible. But first two small lemmas:

Lemma 4.4.2. *Let X,Y be a simplicial complexes and let $X \xrightarrow{[\sigma,\tau]} X_\sigma$ be an elementary $k-$collapse. If $\mathcal{C}\left(Y\right) = \ell$, then there exists a $\left(k+\ell\right)-$collapse*

$$Y * X \overset{k+\ell}{\rightsquigarrow} Y * X_\sigma.$$

Proof. Since $\mathcal{C}(Y) = \ell$, there is a $\ell-$collapse:

$$Y = Y_1 \xrightarrow{[\nu_1,\eta_1]} Y_2 \xrightarrow{[\nu_2,\eta_2]} \ldots Y_{t-1} \xrightarrow{[\nu_{t-1},\eta_{t-1}]} Y_t = \emptyset. \tag{4.1}$$

Denote $Z_i := Y * X - \bigcup_{j=i}^{i-1}\left[\nu_i \uplus \sigma, \eta_i \uplus \tau\right]$ for $i \in [t]$. We will show that for any $i \in [t]$, $\nu_i \cup \sigma$ is a free face in Z_i contained in the unique facet $\eta_i \cup \tau$ of Z_i.

First notice that $\eta_i \cup \tau \in Z_i$. Indeed, otherwise $\eta_i \cup \tau \in [\nu_j \cup \sigma, \eta_j \cup \tau]$ for some $j < i$, but then $\eta_i \in [\nu_j, \eta_j]$ which is a contradiction to the collapse (4.1) of Y. We turn to show that $\nu_j \cup \sigma$ is a free face in Z_i. Pick $\rho_Y \cup \rho_X \in Z_i$ where, $\rho_X \in X$, $\rho_y \in Y$, and both contain $\nu_i \cup \sigma$. Now $\rho_X \subset \tau$ since $\sigma \subset \rho_X \in Y$. In view of $\nu_i \subset \rho_Y$, if $\rho_Y \notin X_i$ then $\rho_Y \in [\nu_j, \eta_j]$ for some $j < i$ and then $\nu_j \cup \sigma = \rho_Y \cup \rho_X \subset \eta_j \cup \tau$, in contradiction with $\rho_Y \cup \rho_X \in Z_i$. Therefore since $\nu_i \subset \rho_Y$, $\rho_Y \in X_i$, and $\rho_Y \subset \eta_i$, we conclude that $\rho = \rho_Y \cup \rho_X \subset \eta_i \cup \tau$, and therefore $\nu_j \cup \sigma$ is free in Z_i

We have established that $\nu_i \cup \sigma$ is a free face of Z_i contained in a unique facet $\eta_i \cup \tau \in Z_i$ and therefore $Z_i \xrightarrow{[\nu_i \uplus \sigma, \eta_i \uplus \tau]} Z_{i+1}$ is an elementary $\left(k+\ell\right)-$collapse, since $|\nu_i \cup \sigma| \leq k + \ell$.

Finally, we are left with showing that:

$$Z_t = Y * X - \biguplus_{j<t}\left[\nu_i \uplus \sigma, \eta_i \uplus \tau\right] = Y * X_\sigma.$$

Indeed, on the one hand if $\rho = \rho_X \cup \rho_y \in Y * X_\sigma$ then $\rho_X \notin [\sigma, \tau]$ and thus $\rho \notin [\nu_i \uplus \sigma, \eta_i \uplus \tau]$ for any i so $\rho \in Z_t$. On the other hand, if $\rho \notin X_\sigma * Y$ but $\rho \in X * Y$, then $\rho_X \in [\sigma, \tau]$. Therefore there exists j such that $\rho_Y \in [\nu_j, \eta_j]$ and hence $\rho \in [\nu_j \cup \nu, \tau_j \cup \eta]$ which gives us that $\rho \notin Z_t$. Combining both sides we get that $Z_t = Y * X_\sigma$. ■

Lemma 4.4.3. *Let X be a $\ell-$star-collapsible complex and Y be $k-$star-collapsible complex, then $X * Y$ is a $\left(k+\ell\right)-$star-collapsible complex.*

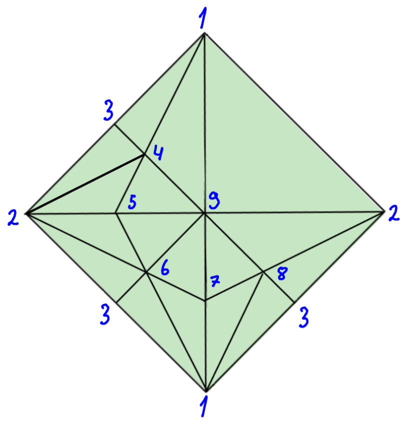


Figure 4.1: 2−star-collapsible but not strongly 2−collapsible

Proof. We argue this by induction on the size of $|V_Y \uplus V_X|$. For $|V_Y \uplus V_X| = 1$ the simplicial complex consists of a single vertex and hence we are done. Now assume that $|V_Y \uplus V_X| = n$. Since X is ℓ−star-collapsible there is a vertex $v \in V_X$ which has the ℓ−collapse:

$$X = X_1 \xrightarrow{[\sigma_1,\tau_1]} X_2 \xrightarrow{[\sigma_2,\tau_2]} \dots X_{t-1} \xrightarrow{[\sigma_{t-1},\tau_{t-1}]} X_t = St\,(v, X)\,.$$

Since $Y \in \mathcal{C}^\ell$ we have the ℓ−collapse:

$$Y = Y_1 \xrightarrow{[\nu_1,\eta_1]} Y_2 \xrightarrow{[\nu_2,\eta_2]} \dots Y_{s-1} \xrightarrow{[\nu_{s-1},\eta_{s-1}]} Y_s = \emptyset.$$

Using Lemma 4.4.2 repeatedly we get:

$$Y * X = Y * X_1 \overset{k+\ell}{\rightsquigarrow} Y * X_2 \overset{k+\ell}{\rightsquigarrow} \dots Y * X_{t-1} \overset{k+\ell}{\rightsquigarrow} Y * X_t = Y * St\,(v, X)\,.$$

But $Y * St\,(v, X) = St\,(v, X * Y)$, and from Proposition 4.2.3 and Lemma 4.2.6 $St\,(v, X * Y) \overset{k+\ell}{\rightsquigarrow} \emptyset$.

Since

$$X\,[V_X - \{v\}] * Y = X * Y\,[V_Y \uplus V_X - \{v\}]\,,$$

$X\,[V_X - \{v\}]$ is ℓ−star-collapsible and Y is k−star-collapsible, by induction $X * Y\,[V_Y \uplus V_X - \{v\}]$ is $(k + \ell)$−star-collapsible. Therefore we get that $X * Y$ is $(k + \ell)$−star-collapsible. ■

The following example describes a family of simplicial complex for which there is a difference between strong collapsibility and star collapsibility.

Example 4.4.4. Let X be the simplicial complex in Figure 4.1, which is taken from [Tan11]. The only 2−collapsible edge is $\{1, 2\}$ and hence there can not be a 2−collapse $X \overset{2}{\rightsquigarrow} St\,(\{1\}, X)$, thus X is not strongly 2−collapsible. But since $\dim(X) = 2$, X is strongly 3−collapsible. On the other hand X is 2−star-collapsible, where one possible order on the vertices is $(6, 9, 7, 5, 1, 8, 2, 3, 4)$. Also note that X is 2−collapsible.

Denote by $Y_i := Y_{i-1} * X$ where $Y_1 := X$. By Lemma 4.2.3 we know that $\mathcal{C}(Y_i) = 2i$ and by Lemma 4.4.3 we get that Y_i is $2i-$star-collapsible. But since there is only one free face of size $2i \in Y_i$, we get that Y_i is not strongly $(2i)-$collapsible.

4.5 $1-$star-collapsibility

In this section we will show that a $1-$collapsible simplicial complex is also $1-$star-collapsible. This will give us that Proposition 4.3.2 is true for $1-$collapsible simplicial complexes. We will also show an 'inverse' of Proposition 4.3.2. This will be done by first 'translating' the Lemma 2.2.3 from the language of chordal graphs into the one of $1-$collapsible complexes:

Lemma 4.5.1. *If X is a 1-collapsible simplicial complex which is not a simplex, then there are at least two free vertices $v, u \in V$, such that $\{v, u\} \notin X$.*

Proof. First recall that by Proposition 3.3.1 since X is 1-collapsible it is a flag complex of a chordal graph. Denote its chordal graph by G_X. Since X is not a simplex G_X is not a complete complex. According to Lemma 2.2.3 if a chordal graph is not the complete graph, then it has at least two simplicial vertices $v, u \in V(G_X)$, such that $\{v, u\} \notin E(G_X)$. Since every simplicial vertex in G_X is free in X, we have two free vertices in X for which $\{v, u\} \notin X$. ■

Now we are ready to prove:

Lemma 4.5.2. *Let X be a 1-collapsible simplicial complex on V. Then X is $1-$star-collapsible.*

Proof. It is enough show to that for any 1-collapsible simplicial complex there is a vertex v such that $X \overset{1}{\rightsquigarrow} St(v, X) \overset{1}{\rightsquigarrow} \emptyset$. Indeed due to Corollary 4.1 we have that $\mathcal{C}(X[V - \{v\}]) \leq 1$ and then we can apply the claim repeatedly on $X[V - \{v\}]$.

We prove a bit more. We prove that any vertex that is free in X can be the vertex we are looking for. We argue it by induction on the size of the vertex set $|V|$ of the simplicial complex X. For $|V| = 1$, it is immediate since it is a single vertex. Now let X be simplicial complex with $|V| = n$ vertices. If $X = \Delta_n$ we are done since any vertex $v \in V$ is free and for each $St(v, X) = \Delta_n = X$. Assume that X is not a simplex. Pick a free vertex $v \in V$, by Lemma 4.5.1 we have two free vertices $u, w \in V$ with $\{u, w\} \notin X$. Either u or w is not in the star of v, since v is free (which implies that $St(v, X) = \Delta_k$). Assume without loss of generality that $u \notin St(v, X)$, and denote $X_u := X[V - \{u\}]$, the simplicial complex after the collapse of u. Since $V(X_u) = n - 1$ and v is still a free vertex with its star in X_u, by the induction assumption there is $X_u \overset{1}{\rightsquigarrow} St(v, X)$. Combining the $1-$collapses we get

$$X \overset{u}{\rightarrow} X_u \overset{1}{\rightsquigarrow} St(v, X)$$

This lemma gives us that Proposition 4.3.2 is true for 1−collapsible complexes.
For 1−collapsible we can do a bit more. The first thing that we want to show is that a 1−collapsible complex does not "feel" elementary 2−collapse:

Lemma 4.5.3. *Let X be a 1-collapsible simplicial complex, and $\sigma \in X(1)$ a free face with a maximal face τ. Then the simplicial complex obtained by collapsing σ, i.e. $X \xrightarrow{[\sigma,\tau]} Y$, is still 1-collapsible.*

Proof. We argue the lemma by induction on the size of $|V|$. For $|V| = 2$ the only complex for which the conditions of the lemma apply is Δ_1. It has only one $\sigma \in X(1)$ and collapsing it gives us the complex containing two disconnected points and is therefore 1-collapsible.

Denote the vertices set by $V := [n]$. If $X = \Delta_{n-1}$ and $\sigma = \{i, j\}$ then Y's facet are $\{[n] - \{i\}, [n] - \{j\}\}$ hence the vertices i and j are free. Collapsing both of them would give us

$$Y \xrightarrow{[\{i\},[n]-\{j\}]} Y[V - \{i\}] \xrightarrow{[\{j\},[n]-\{i\}]} Y[V - \{i,j\}] = \Delta_{n-3},$$

hence Y is still 1−collapsible.

If $X \neq \Delta_{n-1}$ then by Lemma 4.5.1 there are two free vertices $v, u \in V$ for which $\{v, u\} \notin X$. In particular, at least one of them is not in τ. We can assume without loss of generality that $u \notin \tau$. Denote u's unique facet by η, which is also not contained in τ. Therefore $\eta \in Y$ and u is free in Y. By taking the following elementary 1−collapses:

$$\begin{array}{c} X \xrightarrow{[\{u\},\eta]} \hat{X} := X[V - \{u\}] \\ Y \xrightarrow{[\{u\},\eta]} \hat{Y} := Y[V - \{u\}] \end{array},$$

According to Corollary 4.1 $\mathcal{C}\left(\hat{X}\right) \leq 1$. Note that σ is still free in $\hat{X}$ with the unique facet $\tau \in \hat{X}$ and hence $\hat{X} \xrightarrow{[\sigma,\tau]} \hat{Y}$ is a 2−collapse. Therefore by induction $\hat{Y}$ is 1−collapsible and since $Y \xrightarrow{[\{u\},\eta]} \hat{Y}$, Y is also 1−collapsible. ■

Using this lemma we get:

Proposition 4.5.4. *Let X be a simplicial complex with the vertices V. X is 1-collapsible if and only if there is a 2-collapse $\Delta_{|V|-1} \overset{2}{\rightsquigarrow} X$.*

Proof. Denote $|V| = n$. First since $\mathcal{C}(X) = 1$, we get from Lemma 4.5.2 that X is also 1−star-collapsible and from Lemma 4.3.2, we conclude that $\Delta_{n-1} \overset{2}{\rightsquigarrow} X$.

On the other hand assume that there is a 2−collapse:

$$\Delta_{n-1} \xrightarrow{[\sigma_1,\tau_1]} Y_1 \xrightarrow{[\sigma_2,\tau_2]} ...Y_{m-1} \xrightarrow{[\sigma_m,\tau_m]} Y_m = X.$$

We will show that X is 1−collapsible. Since Δ_{n-1} is 1−collapsible, by Lemma 4.5.3 Y_1 is also 1−collapsible. Assume that Y_{k-1} is 1−collapsible. Therefore by Lemma 4.5.3, we get that Y_k is also $1-$collapsible. Finally it is true for all $i \in [m]$, since $X = Y_m$ we obtain that X is also $1-$collapsible. ■

Chapter 5

Consistent Measures & Extensions

5.1 Intoduction

In this chapter we show a nice application of $1-$collapsibility, which was given by Vorob'ev in [Vor62]. In his work the concept of $1-$collapsibility did not exist yet, so the following is a retelling of his ideas.

Let X be an arbitrary set, and $\mathscr{B}$ be a family of σ-algebras on X. For each $\mathcal{B} \in \mathscr{B}$, let $\mu_{\mathcal{B}}$ be a probability measure on the measurable space $(X, \mathcal{B})$. We denote this family of measures by $\mu_{\mathscr{B}}$. The measures $\mu_{\mathcal{B}_1}, \mu_{\mathcal{B}_2} \in \mu_{\mathscr{B}}$ will be called *consistent* if $\forall A \in \mathcal{B}_1 \cap \mathcal{B}_2,\ \mu_{\mathcal{B}_1}(A) = \mu_{\mathcal{B}_2}(A)$. We call the family $\mu_{\mathscr{B}}$ *consistent* if every two measures are consistent. Given a family of sets $\mathcal{F}$ denote by $\sigma(\mathcal{F})$ the $\sigma-$algebra generated by the family $\mathcal{F}$. We call a consistent family $\mu_{\mathscr{B}}$ *extendable* if there exists a measure μ on the measurable space $(X, \sigma(\cup_{\mathcal{B} \in \mathscr{B}} \mathcal{B}))$ such that the $\mu \cup \mu_{\mathscr{B}}$ is a consistent family.

Let X be a simplicial complex on the vertex set V. For each $v \in V$ let S_v be a finite set, and $\mathcal{B}_v$ a $\sigma-$algebra on S_v, so that $(S_v, \mathcal{B}_v)$ is a measurable space. Denote $S = \prod_{v \in V} S_v$. For any $\tau \subset V$ we define a measurable space $(S, \mathcal{B}_\tau)$ where

$$\mathcal{B}_\tau = \left\{ A \times \prod_{v \in V \setminus \tau} S_v \ :\ A \in \sigma\left(\prod_{v \in \tau} \mathcal{B}_v\right)\right\}.$$

Finally, denote by μ_τ a measure on the measurable space $(S, \mathcal{B}_\tau)$. A family $\{\mu_\tau\}_{\tau \in X}$ is extendable if there exists a consistent measure μ on $(S, \mathcal{B}_V)$.

We are interested in the question: Whether any consistent family $\{\mu_\tau\}_{\tau \in X}$ on a simplicial complex X is extendable. This question was raised by Vorob'ev in [Vor62], where he found that :

Theorem 5.1 (Vorob'ev [Vor62])**.** *A simplicial complex X is $1-$collapsible if and only if any consistent family of measures $\{\mu_\tau\}_{\tau \in X}$ on X is extendable.*

We aim to give a proof for Theorem 5.1 using the concept of 1−collapsibility, and to extend it. We denote by $\mathcal{E}$ the family of simplicial complexes for which every consistent family of measures is extendable.

First, we give two examples of families of simplicial complexes which are not in $\mathcal{E}$.

Example 5.1.1. For $n \geq 3$ let Z_n be the simplicial complex on vertices $[n]$ with faces:

$$Z_n = \{\{1,2\},\{2,3\} \dots \{n-1,n\},\{n,1\}\}\,.$$

Set the probability space for each vertex $v \in [n]$ to be $S_v = \{0,1\}$, and set the σ-algebra $\mathcal{B}_v = 2^{S_v}$. We split the discussion into two cases according to the parity of n.

Let n be odd. Denote Z_n's faces by $\sigma_i := \{i, i+1(\mod n)\}$ for any $i \in [n]$. Set the probability measure μ_{σ_i} to be:

$$\mu_{\sigma_i}\left(S_v^{i-1} \times (0,1) \times S_v^{n-i-1}\right) = \mu_{\sigma_i}\left(S_v^{i-1} \times (1,0) \times S_v^{n-i-1}\right) = 0.5\,.$$

This is a consistent family of measures. Assume for contradiction that the family is extendable. Then there exists a probability measure μ on $(S, \mathcal{B}_V)$ which is consistent with the given family of measures. Therefore $\forall i \in [n]$

$$\mu\left(S_v^{i-1} \times (0,0) \times S_v^{n-i-1}\right) = \mu\left(S_v^{i-1} \times (1,1) \times S_v^{n-i-1}\right) = 0\,,$$

since it is consistent with μ_{σ_i}, and:

$$\mu_{\sigma_i}\left(S_v^{i-1} \times (0,1) \times S_v^{n-i-1}\right) + \mu_{\sigma_i}\left(S_v^{i-1} \times (1,0) \times S_v^{n-i-1}\right) = 1\,.$$

We get that $\mu(x) = 0$ for any $x \in S$ containg two consecutive 0's or 1's. But since n is odd every x contains either two consecutive 0's or 1's. Therefore we get a contradiction to our assumption, and $Z_n \notin \mathcal{E}$ for any odd n.

Let n be even. For all $i \in [n-1]$ we set the probability measures:

$$\mu_{\sigma_i}\left(S_v^{i-1} \times (0,1) \times S_v^{n-i-1}\right) = \mu_{\sigma_i}\left(S_v^{i-1} \times (1,0) \times S_v^{n-i-1}\right) = 0.5\,,$$

and

$$\mu_{\sigma_n}\left(\{0\} \times S_v^{n-2} \times \{0\}\right) = \mu_{\sigma_n}\left(\{1\} \times S_v^{n-2} \times \{1\}\right) = 0.5\,.$$

Again, this is a consistent family of measures, and we assume for contradiction that μ is its extension. Note that for any $i \in [n-1]$,

$$\mu\left(S_v^{i-1} \times (0,0) \times S_v^{n-i-1}\right) = \mu\left(S_v^{i-1} \times (1,1) \times S_v^{n-i-1}\right) = 0.$$

But since in the series $(\{0\} \times S_v^{n-2} \times \{0\})$ there are at least two consecutive 1's or 0's we have:

$$\mu_{\sigma_n}\left(\{0\} \times S_v^{n-2} \times \{0\}\right) = \mu\left(\{0\} \times S_v^{n-2} \times \{0\}\right) = 0.$$

We get a contradiction to our assumption, since $\mu_{\sigma_n}\left(\{0\}\times S_v^{n-2}\times\{0\}\right)\neq 0$. Therefore $Z_n\notin\mathcal{E}$ for any $n\in\mathbb{N}$.

Example 5.1.2. In [Vor62] the author shows that $\partial\Delta_n\notin\mathcal{E}$ for any $n\geq 3$.

5.2 Proof of Vorob'ev's Theorem

First, we show an equivalent version of Corollary 4.1 for simplicial complexes in $\mathcal{E}$:

Lemma 5.2.1. *Let $X\in\mathcal{E}$ have the vertex set V. Then $X[A]\in\mathcal{E}$ for any $A\subset V$.*

Proof. We first show the lemma for $A=V\setminus\{u\}$ for any $u\in V$. Let $\{\mu_\tau\}_{\tau\in X[A]}$ be a consistent family of measures on $X[A]$. For any vertex $v\in A$, denote by S_v the set associated to it, and by B_v its $\sigma-$algebra. We now construct a consistent family for X. For any $v\in A$, let the associated measurable space be $(\tilde{S}_v,\tilde{\mathcal{B}}_v)$ be $(S_v\mathcal{B}_v)$ and for $\{u\}$ set $\tilde{S}_u=\{u\}$ and $\tilde{\mathcal{B}}_u=2^{S_u}$. Now for any $\tau\in X[A]$ and any $\tilde{B}\in\tilde{\mathcal{B}}_\tau$, note that $\tilde{B}=B\times\tilde{S}_u$ for some $B\in\mathcal{B}_\tau$, and set $\nu_\tau\left(\tilde{B}\right):=\mu_\tau(B)$. For any face $\tau\in X$ containing the vertex u, we define the measure to be:

$$\tilde{B}\in\mathcal{B}_\tau=\left\{B\times\tilde{S}_u\times\prod_{v\in V\setminus\tau}\tilde{S}_v\ :\ B\in\mathcal{B}_{\tau-\{u\}}\right\},$$
$$\nu_\tau(\tilde{B})=\nu_\tau(B\times\tilde{S}_u)=\mu_{\tau-\{u\}}(B)$$

One can verify that the family $\{\nu_\tau\}_{\tau\in X}$ is a consistent family of probability measures on X. By assumption it can be extended to some measure ν_X to it. We define a measure $\mu_{X[A]}$ on $(S,\mathcal{B}_A)$ by setting for any $B\in\mathcal{B}_A$:

$$\mu_{X[A]}(B)=\nu_X(B\times S_u).$$

For any $\tau\in X[A]$, we get that μ_τ and $\mu_{X[A]}$ are consistent, since we get that for any $B\in\mathcal{B}_\tau$:

$$\mu_\tau(B)=\nu_\tau(B\times S_u)=\nu_X(B\times S_u)=\mu_{X[A]}(B).$$

Therefore $\mu_{X[A]}$ is consistent with the family $\{\mu_\tau\}_{\tau\in X}$ on $X[A]$ and hence $X[A]\in\mathcal{E}$.

Let $A=\{v_i\}_{i=1}^n\subset V$, and denote by $X_j:=X\left[V-\{v_i\}_{i=1}^j\right]$ for $j\in[n]$. From what we showed above, if $X_j\in\mathcal{E}$ then so does X_{j+1}. Therefore by using the argument repeatedly we get that $X[A]=X_n\in\mathcal{E}$. ■

Using the previous lemma we can now show the first side of Theorem 5.1:

Proposition 5.2.2 (Vorob'ev [Vor62])**.** *If $X\notin\mathcal{C}^1$ then $X\notin\mathcal{E}$.*

Proof. Recall that by Proposition 3.3.1 $X\in\mathcal{C}^1$ if and only if it is a clique complex of a chordal graph. So, if X is not $1-$collapsible it has one of the following properties:

Case 1 $X \cong X'$, where X' is a clique complex of graph G which is not chordal :

If G is not chordal it has a cycle C_n of length $n > 3$ with no chord between any two of its vertices. Using Lemma 5.2.1 we get that if X' is in $\mathcal{E}$ then $X'[C_n]$ is in $\mathcal{E}$. But since $X'[C_n] \cong Z_n$ is the simplicial complex from Example 5.1.1, which we know is not in $\mathcal{E}$, we get that $X' \notin \mathcal{E}$.

Case 2 X is not isomorphic to a clique complex:

Like in Lemma 3.3.3 we can find $A \subset V$ with $|A| \geq 3$, for which $X[A] \cong \partial\Delta_{|A|-1}$. By Example 5.1.2 we know that $\partial\Delta_{|A|-1} \notin \mathcal{E}$, and using Lemma 5.2.1 we get that $X[A]$ is also not in $\mathcal{E}$.

Combining Cases 1 and 2, we get that $X \notin \mathcal{E}$. ■

Let Y be a *subcomplex of* X with a consistent family of measures $\{\mu_\tau\}_{\tau\in Y}$. We say that the family $\{\mu_\tau\}_{\tau\in Y}$ is *extendable to* X if there exist a consistent family of measures $\{\nu_\tau\}_{\tau\in X}$, such that $\mu_\tau = \nu_\tau$ for any $\tau \in Y$. Using this definition, the consistent family of measures $\{\mu_\tau\}_{\tau\in X}$ on X is extendable if and only if it is extendable to $\Delta_{|V|-1}$.

Lemma 5.2.3. *Let X be a simplicial complex with the vertex set V, and let $X \xrightarrow{[\sigma,\tau]} Y$ be an elementary* 2−collapse. *Then any consistent family of measures on Y is extendable to X.*

Proof. Let $\{\mu_f\}_{f\in Y}$ be a consistent family of probability measures on Y. We want to define a family of probability measures $\{\nu_f\}_{f\in X}$ on X that extends $\{\mu_f\}_{f\in Y}$. To define the family $\{\nu_f\}_{f\in X}$ on X it would be enough to define the probability measure ν_τ on $(S, \mathcal{B}_\tau)$:

Denote the following faces of X, $\sigma = \{v, u\}$ for $v, u \in V$, $\tau_i = \tau - \{i\}$ for $i \in \sigma$, and $C = \tau\backslash\sigma$. For any face $f \in X$ denote $x_f^* = x_f \times S_{V\backslash f}$ where $x_f \in S_f := \prod_{i\in f} S_i$. To define a measure on $(S, \mathcal{B}_\tau)$, it is enough to define it on the atoms x_τ^* for any $x_\tau \in S_\tau$, as the space in finite. Note that $\tau_v, \tau_u \in Y$ and $x_\tau := \left(x_{\{v\}}, x_{\{u\}}, x_C\right)$. We define the measure ν_τ for any $x_f^* \in \mathcal{B}_f$ to be:

$$\nu_\tau\left(x_\tau^*\right) := \begin{cases} \frac{\mu_{\tau_v}((x_v,S_u,x_C)^*)\mu_{\tau_u}((S_v,x_u,x_C)^*)}{\mu_C((S_u,S_v,x_C)^*)} & \mu_C\left((S_u, S_v, x_C)^*\right) \neq 0 \\ 0 & else \end{cases}.$$

ν_τ is a probability measure since (for brevity we drop the $*$ notation):

$$
\begin{aligned}
\sum_{x\in S_\tau} \nu_\tau\left(x_\tau\right) &= \sum_{x\in S_\tau} \frac{\mu_{\tau_v}\left(x_v, S_u, x_C\right)\mu_{\tau_u}\left(S_v, x_u, x_C\right)}{\mu_C\left(S_v, S_u, x_C\right)} \\
&= \sum_{x_C\in S_C}\sum_{x_v\in S_v}\sum_{x_u\in S_u} \frac{\mu_{\tau_v}\left(x_v, S_u, x_C\right)\mu_{\tau_u}\left(S_v, x_u, x_C\right)}{\mu_C\left(S_v, S_u, x_C\right)} \\
&= \sum_{x_C\in S_C} \frac{1}{\mu_C\left(S_v, S_u, x_C\right)}\left[\sum_{x_v\in S_v}\mu_{\tau_v}\left(x_v, S_u, x_C\right)\left(\sum_{x_u\in S_u}\mu_{\tau_u}\left(S_v, x_u, x_C\right)\right)\right] \\
&= \sum_{x_C\in S_C} \frac{1}{\mu_C\left(S_v, S_u, x_C\right)}\left[\mu_{\tau_v}\left(S_v, S_u, x_C\right)\mu_{\tau_u}\left(S_v, S_u, x_C\right)\right] = (*)
\end{aligned}
$$

Since μ_{τ_v} and μ_{τ_u} are consistent with μ_C we can conclude that:

$$
(*) = \sum_{x_C\in S_C} \frac{\left(\mu_C\left(S_v, S_u, x_C\right)\right)^2}{\mu_C\left(S_v, S_u, x_C\right)} = 1
$$

Now we use ν_τ to define a family of measures on X. Again, to define the measure on $(S, \mathcal{B}_f)$ for $f\in X$ it is enough to define it on the atoms x_f^* for any $x_f\in S_f$. Hence for any $x_f^*\in\mathcal{B}_f$ we define the probabilities:

$$
\nu_f\left(x_f^*\right) := \begin{cases} \mu_f\left(x_f^*\right) & f\in X\cap Y \\ \nu_\tau\left(x_f^*\right) & f\in[\sigma,\tau] \end{cases}.
$$

For any $f\in Y$ we have $\nu_f = \mu_f$. Hence all that is left to show is that the family $\{\nu_f\}_{f\in X}$ is consistent.

Given two faces $f_1, f_2\in X$, if $f_1\cap f_2=\emptyset$ then $\mathcal{B}_{f_1}\cap\mathcal{B}_{f_2}=\{S,\emptyset\}$. Since $\nu_{f_1}(S) = 1 = \nu_{f_2}(S)$, the measures are consistent. Thus from now on we assume that $f_1\cap f_2\neq\emptyset$. Denote $\eta := f_1\cap f_2$ and split the remainder of the proof into 3 cases:

Case 1 $f_1, f_2\in X\cap Y$: Since ν_{f_i} is equal to μ_{f_i} for $i\in[2]$ and μ_{f_1} is consistent with μ_{f_2}, ν_{f_1} is consistent with ν_{f_2}.

Case 2 $f_1, f_2\in[\sigma,\tau]$: Notice that $\mathcal{B}_{f_1}\cap\mathcal{B}_{f_2}=\mathcal{B}_\eta$ and hence it is enough to show that μ_{f_1} and μ_{f_2} agree on any atoms x_η^* for $x_\eta\in S_\eta$. This is true since:

$$
\nu_{f_1}\left(x_\eta^*\right) = \nu_\tau\left(x_\eta^*\right) = \nu_{f_2}\left(x_\eta^*\right).
$$

Case 3 $f_1\in[\sigma,\tau]$, $f_2\in Y$: Since $\tau\notin Y$ we know that $\eta\subset\tau_v$ or $\eta\subset\tau_u$. Assume without loss of generality that $\eta\subset\tau_v$. Then $\eta\cap\tau_u\subset C\,(=\tau_v\cap\tau_u)$, and since μ_{τ_u} is consistent with μ_C and $x_{\eta-\{v\}}^*\in\mathcal{B}_C$ for any $x_{\eta-\{v\}}\in S_{\eta-\{v\}}$, we get:

$$
\mu_{\tau_u}\left(\left(S_v, S_u, x_{\eta-\{v\}}\right)^*\right) = \mu_C\left(\left(S_v, S_u, x_{\eta-\{v\}}\right)^*\right).
$$

Thus for any $x_\eta \in S_\eta$ we get:

$$\nu_{f_1}\left(x_\eta^*\right) = \nu_\tau\left(x_\eta^*\right) = \frac{\mu_{\tau_v}\left(x_\eta^*\right)\mu_{\tau_u}\left(\left(S_v, S_u, x_{\eta-\{v\}}\right)^*\right)}{\mu_C\left(\left(S_v, S_u, x_{\eta-\{v\}}\right)^*\right)}$$
$$= \frac{\mu_{\tau_v}\left(x_\eta^*\right)\mu_C\left(\left(S_v, S_u, x_{\eta-\{v\}}\right)^*\right)}{\mu_C\left(\left(S_v, S_u, x_{\eta-\{v\}}\right)^*\right)} = \mu_{\tau_v}\left(x_\eta^*\right) = (*)\,.$$

Since μ_{τ_v} and ν_{f_2} are consistent on $\mathcal{B}_\eta$ from Case 2 and the defenitino of ν_{f_1}, we get:

$$\nu_{f_1}\left(x_\eta^*\right) = (*) = \mu_{\tau_v}\left(x_\eta^*\right) = \nu_{f_2}\left(x_\eta^*\right).$$

We get that the family $\{\mu_f\}_{f\in X}$ is consistent, which finishes the proof. ■

Using the previous lemma we get that:

Proposition 5.2.4. *Let X, Y be simplicial complexes. If $X \overset{2}{\rightsquigarrow} Y$, then any consistent family of measures on Y is extendable to X.*

Proof. Denote the given 2−collapse by:

$$X = X_1 \xrightarrow{[\sigma_1,\tau_1]} X_2 \xrightarrow{[\sigma_2,\tau_2]} \ldots X_{t-1} \xrightarrow{[\sigma_{t-1},\tau_{t-1}]} X_t = Y.$$

From Lemma 5.2.3 we get that any consistent family on X_i is extendable to X_{i-1} for any $i \in [t]$. Therefore one step at a time we can extend every consistent family on Y to X. ■

Now using results from the previous chapters we can prove the Vorob'ev's theorem:

Proof of Theorem 5.1. If $X \in \mathcal{C}^1$ then according to Proposition 4.5.4 there is a 2−collapse $\Delta_{|V|-1} \overset{2}{\rightsquigarrow} X$. By Proposition 5.2.4 any family of consistent measures on X is extendable to $\Delta_{|V|-1}$ and hence $X \in \mathcal{E}$. On the other hand if $X \notin \mathcal{C}^1$, then by Proposition 5.2.2 $X \notin \mathcal{E}$. So we conclude that $\mathcal{C}^1 = \mathcal{E}$. ■

Chapter 6

Concluding Remarks

This thesis studied various aspects of $d-$collapsibility of simplicial complexes. Our results give rise to a number of additional questions, and conjectures. In particular:

1. In Theorem4.2 we have shown that for any complexes X, Y it holds that

$$C\left(X \cap Y\right) \leq \mathcal{C}\left(X\right) + \mathcal{C}\left(Y\right).$$

   We also obtained an analogues result for unions, i.e

$$C\left(X \cup Y\right) \leq \mathcal{C}\left(X\right) + \mathcal{C}\left(Y\right) + 1, \tag{6.1}$$

   provided that X is $\mathcal{C}\left(X\right)-$star-collapsible.
   It would be interesting to establish inequality (6.1) in its full generality.

2. One possible direction for proving inequality (6.1) would be to show the following:

   Conjecture 6.0.1. Let X be a simplicial complex with the vertices set V. If X is $d-$collapsible, than there is a $(d+1)$-collapse $\Delta_{|V|-1} \overset{d+1}{\rightsquigarrow} X$.

   We have verified the conjecture when X is $d-$star-collapsible. Hence it would be enough to show that any $d-$collapsible simplicial complex is $d-$star-collapsible simplicial complex.

3. In [MK07] Meshulam and Kalai, show more than just a bound on the Leray number of a union and intersection of simplicial complex:
   Let X be a simplicial complex with the vertex set V. Suppose $V = \bigcup_{i=1}^{n} V_i$ is a partition of V such that the induced subcomplexes $X[V_i]$ are all $0-$dimensional. Let π denote the projection of X into the $(m-1)-$simplex with the vertex set $[m]$ given by $\pi(v) = i$ if $v \in V_i$.

   Theorem 6.1 ([MK07]). *For the complex X, as above, denote by*

$$r = \max\left\{\left|\pi^{-1}\left(\pi\left(\sigma\right)\right)\right| : \sigma \in X\right\},$$

then

$$\mathcal{L}(\pi(X)) \leq r\mathcal{L}(X) + r - 1. \tag{6.2}$$

We conjecture a similar inequality holds for $d-$collapsibility i.e.

Conjecture 6.0.2. For any complex X, as above, denote by

$$r = \max\left\{\left|\pi^{-1}(\pi(\sigma))\right| : \sigma \in X\right\},$$

then

$$\mathcal{C}(\pi(X)) = r\mathcal{C}(X) + r - 1.$$

This theorem generalizes inequality (6.1) since a union is just the case where for each $i \in [n]$, $|V_i| \leq 2$.

4. In proposition 4.2.3 we have shown that $\mathcal{C}(X * Y) \leq \mathcal{C}(X) + \mathcal{C}(Y)$. It would be interesting to decide whether an equality always holds.

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נסמן ב-μ_τ מידת הסתברות על $(S,\mathcal{B}_\tau)$. המידות הסתברות $\mu_{\tau_1},\mu_{\tau_2}$ נקראות עקביות אם:

$$\forall A\in\mathcal{B}_{\tau_1}\cap\mathcal{B}_{\tau_2},\mu_{\tau_1}(A)=\mu_{\tau_2}(A).$$

למשפחה של מידות הסתברות אנו נקרא עיקבית אם כל זוג מידות במשפחה עקביות. אנו נגיד כי המשפחה $\{\mu_\tau\}_{\tau\in X}$ ניתנת להרחבה אם קיימת מידת הסתברות μ על $(S,\mathcal{B}_V)$ עקבית עם כל מידה במשפחה $\{\mu_\tau\}_{\tau\in X}$. וורובייב הראה כי:

משפט 5.1. יהי X קומפלקס סימפליציאלי. X 1-מטיט אם ורק אם כל משפחה של מידות הסתברות $\{\mu_\tau\}_{\tau\in X}$ עקבית ניתנת להרחבה.

מאחר והמושג של 1-מטיטות לא היה קיים בזמן בו וורובייב הוכיח את המשפט שלו, העבודה שלנו הייתה 'לתרגם' את עבודתו לשפה של מיטוטים. בנוסף, בעזרת התרגום, הצלחנו גם להכליל את המשפט שלו: יהי Y תת-קומפלקס של X ומשפחה של מידות הסתברות $\{\mu_\tau\}_{\tau\in Y}$. נגיד שהמשפחה הזאת ניתנת להרחבה ל-X אם קיימת משפחה עקבית של מידות הסתברות $\{\nu_\tau\}_{\tau\in X}$, כאשר $\mu_\tau=\nu_\tau$ עבור כל $\tau\in Y$. הראנו כי:

טענה 5.2.3. יהיו X,Y קומפלקסים סימפליציאלים. אם קיים 2-מיטוט $X\overset{2}{\rightsquigarrow}Y$, אז כל משפחה של מידות הסתברות $\{\mu_\tau\}_{\tau\in Y}$ עקבית ניתנת להרחבה ל-X.

משפט 6 הוא הכללה של משפט 5. וזאת מכיוון שהראנו כי קומפלקס 1-מטיט הוא קומפלקס 2-כוכב מטיט ולכן מלמה 4 קיים 2-מיטוט $X\overset{2}{\rightsquigarrow}\Delta_{|V|-1}$ וממשפט 6 אנו מקבלים כי כל משפחה על X ניתנת להרחבה ל-$\Delta_{|V|-1}$.

העבודה מאורגנת באופן הבא:

פרק 2 מכיל רקע בגרפים: גרפי חיתוך, גרפי אינטרוולים, וגרפים קורדלים. בנוסף אנו מוכיחים כי גרף אינטרוולי הוא גם קורדלי, ומשפט המראה את ההגדרות השקולות לגרפים קורדלים.

פרק 3 מכיל את ההרחבה הרב ממדית למושגים שהוגדרו בפרק 2: עצב, d-מטיטות, d-יציגות, d.Leray- נוכיח את הקשר בין המושגים, ונראה מספר דוגמאות מעניינות.

פרק 4 עוסק בהוכחת המשפטים 1,2,3 ובדיון על במושג d-כוכב מטיטות.

פרק 5 מכיל הוכחה למשפט וורובייב במונחים של d-מיטוט ואת משפט 6 .

$X*Y$ הוא הקומפלקס הסימפליציאלי:

$$X*Y:=\left\{\sigma_X\cup\sigma_Y\in 2^{V_X\uplus V_Y}:\sigma_X\in X,\sigma_Y\in Y\right\}.$$

והחסם שאנו מקבלים הוא:

טענה 4.2.3. יהי X,Y קומפלקסים סימפליציאלים, אזי:

$$\mathcal{C}\left(X*Y\right)\leq\mathcal{C}\left(X\right)+\mathcal{C}\left(Y\right).$$

עבור אי השיוויון של מספר המיטוט של איחוד קומפלקסים קיבלנו תוצאה יותר מוגבלת. תחילה ניזכר בכך שכוכב של פאה $f\in X$ הוא תת-הקומפלקס $St\left(f,X\right)=\left\{\sigma\in X:f\cup\sigma\in X\right\}$. נקרא לקומפלקס הסימפליציאלי X עם קדקודים V, d-כוכב מטיט אם קיים סדר על קדקודיו $V=\left\{v_i\right\}_{i=1}^{n}$, כך שאם נסמן ב-$X_i:=X\left[V-\left\{v_j\right\}_{j=1}^{i+1}\right]$ אזי המיטוטים הבאים קיימים עבור כל $i\in[n]$:

$$X_{i-1}\overset{d}{\rightsquigarrow}St\left(v_i,X_{i-1}\right)\overset{d}{\rightsquigarrow}\emptyset.$$

עם ההגדרה החדשה אנו מקבלים את החסם הבא:

משפט 4.3. יהי X,Y קומפלקסים סימפליציאלים. אם Y הוא $\mathcal{C}\left(Y\right)$-כוכב מטיט אזי:

$$\mathcal{C}\left(X\cup Y\right)\leq\mathcal{C}\left(Y\right)+\mathcal{C}\left(X\right)+1.$$

השאלה המתבקשת היא האם קומפלקס d-מטיט הוא d-כוכב מטיט. עבור $d=1$ הצלחנו להראות כי קומפלקס 1-מטיט הוא קומפלקס 1-כוכב מטיט. עבור $d\geq 2$ יש לנו מספר סיבות להאמין כי זה נכון, אך לצערנו כרגע זה נישאר בתור השערה. תכונה מעניינת נוספת שמצאנו עבור קומפלקסים d-כוכב מטיטים היא:

טענה 4.3.2. יהי X קומפלקס סימפליציאלי על הקדקודים V, d-כוכב מטיט. אזי קיים $(d+1)$-מיטוט

$$\Delta_{|V|-1}\overset{d+1}{\rightsquigarrow}X,$$

כאשר $\Delta_{|V|-1}$ הוא הסימפלקס ה-$|V|-1$ מימדי.

דבר אחרון עליו נרצה לדבר הוא משפט של וורובייב [Vor62] בנושא של משפחות של מידות הסתברות עקביות הניתנות להרחבה על קומפלקסים סימפליציאלים:
יהי X קומפלקס סימפליציאלי על הקדקודים V. לכל $v\in V$ יהי $\left(S_v,2^{S_v}\right)$ מרחב מדיד סופי. נסמן ב-$S=\prod_{v\in V}S_v$. לכל $\tau\in 2^V$ יהי $\left(S,\mathcal{B}_\tau\right)$ מרחב מדיד סופי, כאשר:

$$\mathcal{B}_\tau=\left\{A\times\prod_{v\in V\setminus\tau}S_v\ :\ A\in\prod_{v\in\tau}S_v\right\}.$$

תקציר

תוצאה קלאסית של הלי [Hel23] אומרת כי בהינתן $\mathcal{K} = \{K_1, \dots K_n\}$ משפחה סופית של קבוצות קמורות ב- $\mathbb{R}^d$ כך ש- $\cap_{i\in I} K_i \neq \emptyset$ עבור כל $I \subset [n]$ מגודל $|I| \leq d+1$, אז $\cap_{i=1}^n K_i \neq \emptyset$. משפט הלי על הכללותיו והרחבותיו הרבות משחקות חלק מרכזי בגאומטריה בדידה ושימושיה. מרכיב חשוב במגוון משפטים מסוג זה הוא d-מיטוט. יהי X קומפלקס סימפליציאלי. יהי $\sigma \in X$, מגודל $|\sigma| \leq d$ המוכל בפאה מקסימלית יחידה τ. d-מיטוט אלמנטרי מוגדר כפעולה על קומפלקס סימפליציאלי $X \xrightarrow{[\sigma,\tau]} X - [\sigma,\tau]$ כאשר $[\sigma,\tau] = \{f \in X : \sigma \subset f \subset \tau\}$. אנו קוראים לקומפלקס X d-מטיט אם קיימת סדרת של d-מיטוטים אלמנטרים המביאה את הקומפלקס X לקומפלקס הריק. נסמן זאת ב- $X \overset{d}{\rightsquigarrow} \emptyset$ ונגדיר את מספר המיטוט של X להיות $\mathcal{C}(X) = \min_{d\in\mathbb{N}} \left\{X \overset{d}{\rightsquigarrow} \emptyset\right\}$. תהא משפחה של קבוצות $\mathcal{K}$. העצב של $\mathcal{K}$, המסומן ב-$N(\mathcal{K})$, הוא הקומפלקס הסימפליציאלי על קבוצת הקדקודים $\mathcal{K}$, והפאות שלו הן כל תתי המשפחות $f \subset \mathcal{K}$ כך ש-$\bigcap_{F\in f} F \neq \emptyset$. הקשר בין d-מיטוט וקמירות הוא משפט של וגנר [Weg75] האומר כי בהינתן $\mathcal{K} = \{K_1, \dots, K_n\}$ משפחה של קבוצות קמורות ב-$\mathbb{R}^d$ העצב $N(\mathcal{K})$ הוא d-מטיט. מושג חשוב ומרכזי נוסף עבורנו הוא מספר Leray של X , המסומן ב-$\mathcal{L}(X)$, המוגדר כ-d המינימלי עבורו

$$\tilde{H}_i(Y, \mathbb{R}) = 0,$$

עבור כל תת-קומפלקס מושרה $Y \subset X$ ו-$i \geq d$. וגנר הראה כי $\mathcal{C}(X) \geq \mathcal{L}(X)$ עבור כל קומפלקס סימפליציאלי X.

בתזה זאת אנו חקרנו היבטים שונים של d-מיטוט ושימושיה. במאמר [MK07] של משולם וקלעי הם מראים כי עבור כל זוג קומפלקסים סימפליציאלים X, Y האי שוויונות הבאים תקפים:

$$\mathcal{L}(X \cap Y) \leq \mathcal{L}(Y) + \mathcal{L}(X)$$
$$\mathcal{L}(X \cup Y) \leq \mathcal{L}(Y) + \mathcal{L}(X) + 1$$

מכיוון שראינו שישנו קשר בין מספר Leray של קומפלקסים סימפליציאלים למספר המיטוט שלהם, אנו ניסינו לשחזר את התוצאות הללו עבור מספרי המטיטות. עבור האי שיוויון על מספר המיטוט של החיתוך הצלחנו להראות כי:

משפט 4.2. יהי X, Y קומפלקסים סימפליציאלים. מספר המיטוט של החיתוך שלהם מקיים:

$$\mathcal{C}(X \cap Y) \leq \mathcal{C}(Y) + \mathcal{C}(X).$$

כמסקנה למשפט 1 אנו מקבילים חסם על $X * Y$. ניזכר כי בהינתן שני קומפלקסים X, Y על הקדקודים V_X, V_Y,

המחקר בוצע בהנחייתו של פרופסור רועי משולם, בפקולטה למתמטיקה.

תודות

אני רוצה להודות למנחה שלי רועי משולם על כל התמיכה והעזרה. תודה לחברי מהפקולטה על כל עזרתם במהלך הכתיבה והתמיכה בי.

אני מודה לטכניון על התמיכה הכספית הנדיבה בהשתלמותי.

מטיטות ושימושיה

חיבור על מחקר

לשם מילוי חלקי של הדרישות לקבלת התואר
מגיסטר למדעים במתמטיקה

איגור חמלניצקי

הוגש לסנט הטכניון — מכון טכנולוגי לישראל
סיון תשע"ח חיפה יוני 2018

מטיטות ושימושיה

איגור חמלניצקי